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ON THE FIRST-PASSAGE AREA FUNCTIONAL OF A TELEGRAPH PROCESS

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ABSTRACT. We study the distribution of the time of the first transition through zero for a one-dimensional piecewise linear random process, when this process starts from $x > 0$. The joint distribution of this time and the area between the time axis and the trajectory of the process until the first return to the origin of coordinates is also considered.

1. Introduction

This paper examines the distribution of the area swept out by a random process, starting from the point $x \neq 0$ and until its first passage to the origin. Our main goal is the study of the statistical properties of such a functional based on the telegraph process.

First-passage phenomena associated with random walks and diffusions, as well as their applications in various fields, are still the focus of interests of physicists and mathematicians. Starting with continuous-time random walks and diffusions, these problems have spread in various directions. Such problems for jump-diffusion processes are being intensively studied, see e.g. [1, 17, 18, 9], for fractional Brownian motion, [10], as well as similar settings for Lévy processes, [3, 6, 21]. There are many applications of such models in physics, [10, 13, 14], in finance, [3, 5, 28], in biology, [22]. See also a detailed review of theoretical results and their applications (including the motions inside a living cell) in [20].

However, it is worth noting that piecewise linear random processes have also come to be used as a core component in various applications, often replacing diffusion processes. There are several reasons for this replacement. In addition to the obvious and generally accepted advantages, traditionally used Wiener processes have some rather unusual from a viewpoint of applications properties, such as non-differentiability and unbounded variation of trajectories, infinite propagation speed, among others. For example, the models mentioned in [2] concerning queueing theory and the economics supply-and-demand will become more adequate to reality if in these models diffusion is replaced by a process with finite speeds.

Distributions of the first-passage time for telegraph processes have been sufficiently studied, see e.g. [8, 11, 19, 23, 25, 26, 27], and a detailed review in the recently published book [29]. The distribution of the process obtained by iterated integration of the telegraph

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process is less known, only some particular problems are solved, see e.g. [16, 24]. In this paper, the main attention is paid to the distribution of the first passage area, which corresponds to iterated integration of the telegraph signal.

We recall definitions and properties of a telegraph process in Section 2. In Section 3, we provide integral and differential equations for the Laplace transforms of the first passage time and the first passage area associated with the telegraph process. Section 4 provides a detailed account of the solutions to these equations. The moments are studied in Section 5, and in Section 6 we analyse these results under diffusion scaling.

2. Telegraph process

Let $X = X^x(t)$, $t \geq 0$, be the position of a particle moving in a line, starting from point x, with constant velocities c_0, c_1, $c_0 > c_1$, alternating in Poisson times, i.e.

$$X^x(t) = x + \int_0^t c_{\varepsilon(s)}\mathrm{d}s. \tag{2.1}$$

Here $\varepsilon = \varepsilon(s) \in \{0,1\}$ is a two-state Markov process with alternating switching intensities λ_0, λ_1, defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$:

$$\mathbb{P}_i\{\varepsilon(t+\mathrm{d}t) = i\} = 1 - \lambda_i \mathrm{d}t + o(\mathrm{d}t), \qquad \mathrm{d}t \to 0, \qquad i \in \{0,1\},$$

where $\mathbb{P}_i$ denotes the conditional probability $\mathbb{P}_i\{A\} := \mathbb{P}\{A \mid \varepsilon(0) = i\}$, $i \in \{0,1\}$.

We call the process X, (2.1), a telegraph process.

Distributions associated with the process $X = X^x(t)$ can be represented explicitly using the following notations,

$$\begin{aligned} \xi_0(t,x) = \frac{x - c_1 t}{c_0 - c_1}, \qquad \xi_1(t,x) = t - \xi_0 = \frac{c_0 t - x}{c_0 - c_1}, \\ \theta(t,x) = \frac{\exp(-\lambda_0 \xi_0(t,x) - \lambda_1 \xi_1(t,x))}{c_0 - c_1}, \\ 0 < \xi_0(t,x),\ \xi_1(t,x) < t, \qquad c_1 t < x < c_0 t, \end{aligned}$$

and

$$\begin{aligned} \mathscr{I}_0(t,x) &= \sum_{n=0}^{\infty} \frac{z(t,x)^n}{n!^2} = I_0(2\sqrt{z})|_{z=z(t,x)}, \\ \mathscr{I}_1(t,x) &= \sum_{n=0}^{\infty} \frac{z(t,x)^n}{n!(n+1)!} = \frac{I_1(2\sqrt{z})}{\sqrt{z}}|_{z=z(t,x)}, \end{aligned} \tag{2.2}$$

where $z(t,x) = \lambda_0 \lambda_1 \xi_0(t,x) \xi_1(t,x)$, and I_0, I_1 are the modified Bessel function.

The distribution of $X^x(t)$ can be described by the conditional probabilities

$$P_0(t, \mathrm{d}y \mid x) := \mathbb{P}\{X^x(t) \in \mathrm{d}y \mid \varepsilon(0) = 0\} \quad \text{and} \quad P_1(t, \mathrm{d}y \mid x) := \mathbb{P}\{X^x(t) \in \mathrm{d}y \mid \varepsilon(0) = 1\},$$

which are explicitly given by

$$\begin{aligned}P_0(t,\mathrm{d}y\mid x) =&\mathrm{e}^{-\lambda_0 t}\delta_{x+c_0t}(\mathrm{d}y)\\&+\lambda_0\left[\mathscr{I}_0(t,y-x)+\lambda_1\xi_0(t,y-x)\mathscr{I}_1(t,y-x)\right]\theta(t,y-x)\mathbb{1}_{\{x+c_1t<y<x+c_0t\}}\mathrm{d}y,\\P_1(t,\mathrm{d}y\mid x) =&\mathrm{e}^{-\lambda_1 t}\delta_{x+c_1t}(\mathrm{d}y)\\&+\lambda_1\left[\mathscr{I}_0(t,y-x)+\lambda_0\xi_1(t,y-x)\mathscr{I}_1(t,y-x)\right]\theta(t,y-x)\mathbb{1}_{\{x+c_1t<y<x+c_0t\}}\mathrm{d}y,\end{aligned}\tag{2.3}$$

see e.g. [29, Chap.3].

Here and below, by $\delta_a(\mathrm{d}y)$ we denote the Dirac delta-measure at point a, so the first terms in (2.3), $\delta_{x+ct}(\mathrm{d}y)$ correspond to rectilinear ("ballistic") movement without speed switching.

In this paper, we study the distributions of the time $\mathscr{T}(x)$, $x>0$, of the first visit to the origin by the process $X^x(t)$ and the area $\mathscr{A}(x)$ swept by the path of this process up to the time $\mathscr{T}(x)$.

The time $\mathscr{T}(x)$ of first passage of the origin and the first passage area $\mathscr{A}(x)$, $x>0$, are defined by

$$\mathscr{T}(x)=\min\{t>0:\ X^x(t)=0\}\quad\text{and}\quad\mathscr{A}(x)=\int_0^{\mathscr{T}(x)}X^x(s)\mathrm{d}s,\tag{2.4}$$

respectively.

Assume that at least one velocity is negative, since otherwise the process a.s. increases and never reaches the origin, i.e. $\mathscr{T}(x)=\infty$ *a.s.*

It is known that the conditional distributions

$$F_i^{\mathscr{T}}(\mathrm{d}t;x)=\mathbb{P}_i\{\mathscr{T}(x)\in\mathrm{d}t\},\qquad i\in\{0,1\},$$

have different forms for cases of both negative velocities and velocities of opposite signs. See [23, Theorem 3.1] or [29, Theorem 5.10]. For completeness, we recall the explicit formulae.

In the case of both negative velocities, $0>c_0>c_1$, the process $X^x(t)$ a.s. decreases down to the origin, so a distribution with a mass point appears,

$$\begin{aligned}F_0^{\mathscr{T}}(\mathrm{d}t,x)&=\mathrm{e}^{\lambda_0x/c_0}\delta_{-x/c_0}(\mathrm{d}t)\\&-\lambda_0\left[c_1\mathscr{I}_0(t,-x)+\lambda_1c_0\xi_0(t,-x)\mathscr{I}_1(t,-x)\right]\theta(t,-x)\mathbb{1}_{\{-x/c_1<t<-x/c_0\}}\mathrm{d}t,\\F_1^{\mathscr{T}}(\mathrm{d}t,x)&=\mathrm{e}^{\lambda_1x/c_1}\delta_{-x/c_1}(\mathrm{d}t)\\&-\lambda_1\left[c_0\mathscr{I}_0(t,-x)+\lambda_0c_1\xi_0(t,-x)\mathscr{I}_1(t,-x)\right]\theta(t,-x)\mathbb{1}_{\{-x/c_1<t<-x/c_0\}}.\end{aligned}\tag{2.5}$$

If the velocities are of opposite signs, $c_0>0>c_1$, then

$$\begin{aligned}F_0^{\mathscr{T}}(\mathrm{d}t,x)=&\frac{\lambda_0}{\xi_1(t,-x)}\left[x\mathscr{I}_0(t,-x)+c_0\xi_0(t,-x)\mathscr{I}_1(t,-x)\right]\theta(t,-x)\mathbb{1}_{\{t>-x/c_1\}}\mathrm{d}t,\\F_1^{\mathscr{T}}(\mathrm{d}t,x)=&\mathrm{e}^{\lambda_1x/c_1}\delta_{-x/c_1}(\mathrm{d}t)+\lambda_0\lambda_1x\mathscr{I}_1(t,-x)\theta(t,-x)\mathbb{1}_{\{t>-x/c_1\}}\mathrm{d}t.\end{aligned}\tag{2.6}$$

See (2.3) and [29, Sect.5.4].

3. Laplace transform: equations

Let $u(x)=\alpha+\beta x>0,\ \alpha,\beta\geq 0$. In this paper, we study the distribution of the random variable

$$\Xi(x):=\int_0^{\mathscr{T}(x)}u(X(s))\mathrm{d}s=\alpha\mathscr{T}(x)+\beta\mathscr{A}(x). \tag{3.1}$$

We will also consider random variables $\Xi_i(x)$ defined by (3.1) for a given initial state $\varepsilon(0)=i,\ i\in\{0,1\}$.

Let τ_i be the first switching time of the underlying Markov process $\varepsilon(t)$ with the given initial state $\varepsilon(0)=i,\ i\in\{0,1\}$.

We denote by $\phi_0(\tau_0,x)$ and $\phi_1(\tau_1,x)$ the accumulated increments up to the first switching time $\tau=\tau_i$ of $\Xi_0(x)$ and $\Xi_1(x)$, respectively,

$$\phi_0(\tau,x)=\int_0^{\tau}u(x+c_0s)\mathrm{d}s,\qquad \phi_1(\tau,x)=\int_0^{\tau}u(x+c_1s)\mathrm{d}s, \tag{3.2}$$

if $0<\tau<\mathscr{T}(x)$.

The following equalities in distribution hold:

$$\Xi_0(x)\overset{D}{=}\phi_0(t,x)|_{t=x/(-c_0)}\mathbb{1}_{x+c_0\tau<0}+[\phi_0(\tau_0,x)+\Xi_1(x+c_0\tau_0)]\,\mathbb{1}_{\{x+c_0\tau_0>0\}}, \tag{3.3}$$

$$\Xi_1(x)\overset{D}{=}\phi_1(t,x)|_{t=x/(-c_1)}\mathbb{1}_{x+c_1\tau<0}+[\phi_1(\tau_1,x)+\Xi_0(x+c_1\tau_1)]\,\mathbb{1}_{\{x+c_1\tau_1>0\}}. \tag{3.4}$$

Note that by (3.2),

$$\phi_0(\tau,x)=u(x)\tau+\frac{1}{2}\beta c_0\tau^2,\qquad \phi_1(\tau,x)=u(x)\tau+\frac{1}{2}\beta c_1\tau^2. \tag{3.5}$$

It's easy to see that

$$\frac{\partial\phi_0}{\partial x}(\tau,x)=\frac{\partial\phi_1}{\partial x}(\tau,x)=\beta\tau,$$
$$\frac{\partial\phi_0}{\partial \tau}(\tau,x)=u(x)+\beta c_0\tau,\qquad \frac{\partial\phi_1}{\partial \tau}(\tau,x)=u(x)+\beta c_1\tau.$$

Therefore,

$$\begin{aligned} c_0\frac{\partial\phi_0(\tau,x)}{\partial x}-\frac{\partial\phi_0(\tau,x)}{\partial\tau}&=-u(x),\\ c_1\frac{\partial\phi_1(\tau,x)}{\partial x}-\frac{\partial\phi_1(\tau,x)}{\partial\tau}&=-u(x).\end{aligned} \tag{3.6}$$

Further, if the velocity c_i is negative, then $c_i\dfrac{\mathrm{d}\psi_i(x)}{\mathrm{d}x}=-u(x)$, where the values ψ_0 and ψ_1 of the mass points of $\Xi_0(x)$ and $\Xi_1(x)$ are defined by

$$\psi_0(x)=\phi_0(t,x)|_{t=x/(-c_0)},\qquad \psi_1(x)=\phi_1(t,x)|_{t=x/(-c_1)}.$$

This mass point corresponds to a "ballistic" movement without velocity switching until the particle reaches the origin. Otherwise, if the initial velocity is positive, the distribution of the corresponding branch of $\Xi(x)$ is absolutely continuous.

We will use the following matrix notations:

$$\Lambda=\begin{pmatrix}-\lambda_0 & \lambda_0\\ \lambda_1 & -\lambda_1\end{pmatrix},\qquad \mathscr{C}=\begin{pmatrix}c_0 & 0\\ 0 & c_1\end{pmatrix},\qquad \mathrm{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

and

$$\mathscr{M}(q,x)=\mathscr{C}^{-1}(-\Lambda+qu(x/2)\mathrm{I})=\begin{pmatrix}\dfrac{\lambda_0+qu(x/2)}{c_0} & -\dfrac{\lambda_0}{c_0}\\[2ex] -\dfrac{\lambda_1}{c_1} & \dfrac{\lambda_1+qu(x/2)}{c_1}\end{pmatrix}. \tag{3.7}$$

Let $\mathbf{L}=\mathbf{L}(q,x)=(L_0,\ L_1)$ be the Laplace transform of $\Xi(x)$,

$$L_0(q,x)=\mathbb{E}_0\left[\exp(-q\Xi(x))\right],\quad L_1(q,x)=\mathbb{E}_1\left[\exp(-q\Xi(x))\right].$$

Here $\mathbb{E}_i$ is the expectation with respect of conditional probability $\mathbb{P}_i$.

Note that

$$\mathscr{A}(x),\ \mathscr{T}(x)\to+\infty,\ a.s.\text{ if }x\to+\infty.$$

Therefore, the Laplace transform of the random variable $\Xi(x)$ satisfies the condition at infinity:

$$\mathbf{L}(q,x)\to 0,\qquad \text{if } x\to+\infty. \tag{3.8}$$

Theorem 3.1. *The Laplace transform $\mathbf{L}(q,x)$ has a form*

$$\mathbf{L}(q,x)=\exp\left(x\mathscr{M}(q,x)\right)\boldsymbol{\ell}_q,\qquad x\geq 0, \tag{3.9}$$

with the boundary value $\boldsymbol{\ell}_q=(\ell_q^{(0)},\ \ell_q^{(1)})^{\mathrm{T}}$, depending on the signs of parameters c_0,c_1 :

- *if $0>c_0>c_1$, then ℓ_q is the unit column, $\boldsymbol{\ell}_q=\mathbf{1}=(1,\ 1)^{\mathrm{T}}$;*
- *if $c_0>0>c_1$, then $\ell_q^{(1)}=1$ and*

$$\ell_q^{(0)}=\int_0^\infty\lambda_0\exp\left(-(\lambda_0+q\alpha)\tau-qc_0\beta\tau^2/2\right)L_1(q,c_0\tau)\mathrm{d}\tau.$$

Proof. Conditioning on the first speed switch and using (3.3)-(3.4), one can obtain a system of coupled integral equations (cf [4] or [29, Section 5.4]). The form of the equations depends on the signs of the parameters. In the case of both negative velocities, i.e. $0>c_0>c_1$, both equations contain free terms corresponding to ballistic movements without switching. We get

$$L_0(q,x)=\exp\left(\lambda_0x/c_0-q\psi_0(x)\right)+\int\limits_0^{x/(-c_0)}\lambda_0\mathrm{e}^{-\lambda_0\tau}\cdot\exp\left(-q\phi_0(\tau,x)\right)L_1(q,x+c_0\tau)\mathrm{d}\tau, \tag{3.10}$$

$$L_1(q,x)=\exp\left(\lambda_1x/c_1-q\psi_1(x)\right)+\int\limits_0^{x/(-c_1)}\lambda_1\mathrm{e}^{-\lambda_1\tau}\cdot\exp\left(-q\phi_1(\tau,x)\right)L_0(q,x+c_1\tau)\mathrm{d}\tau. \tag{3.11}$$

The first term in (3.3)-(3.4) vanishes if the initial velocity $c_i,\ i\in\{0,1\}$, is positive. Hence, if the velocities are of opposite signs, $c_0>0>c_1$, then (3.11) remains valid, but the first equation, (3.10), corresponding to a positive initial velocity has a simpler form:

$$L_0(q,x)=\int_0^\infty\lambda_0\mathrm{e}^{-\lambda_0\tau}\cdot\exp\left(-q\phi_0(\tau,x)\right)L_1(q,x+c_0\tau)\mathrm{d}\tau. \tag{3.12}$$

The integral equations (3.10)-(3.12) can be converted to differential form.

If $c_0 < 0$, then, using (3.6), we obtain

$$\begin{aligned} c_0 L_0'(q,x) = &(\lambda_0 + qu(x)) \exp(\lambda_0 x/c_0 - q\psi_0(x)) \\ &- \lambda_0 e^{\lambda_0 x/c_0} \exp(-q\psi_0(x)) L_1(0,q) \\ &+ c_0 \int_0^{x/(-c_0)} \lambda_0 e^{-\lambda_0 \tau} \left[\frac{\partial}{\partial x} \exp(-q\phi_0(\tau,x))\right] L_1(q, x + c_0\tau) d\tau \\ &+ \int_0^{x/(-c_0)} \lambda_0 e^{-\lambda_0 \tau} \exp(-q\phi_0(\tau,x)) \left[\frac{\partial}{\partial \tau} L_1(q, x + c_0\tau)\right] d\tau. \end{aligned}$$

Here and below, F' denotes the derivative with respect to the spacial variable x, so that $L_0' = \partial L_0/\partial x$.

Integrating by parts in the latter integral, one can see that

$$\begin{aligned} c_0 L_0'(q,x) = &(\lambda_0 + qu(x)) \exp(\lambda_0 x/c_0 - q\psi_0(x)) \\ &+ \int_0^{x/(-c_0)} \lambda_0 e^{-\lambda_0 \tau} \left[c_0 \frac{\partial}{\partial x} \exp(-q\phi_0(\tau,x)) - \frac{\partial}{\partial \tau} \exp(-q\phi_0(\tau,x))\right] L_1(q, x + c_0\tau) d\tau \\ &+ \lambda_0 \int_0^{x/(-c_0)} \lambda_0 e^{-\lambda_0 \tau} \exp(-q\phi_0(\tau,x)) L_1(q, x + c_0\tau) d\tau - \lambda_0 L_1(q,x), \end{aligned}$$

which by virtue of (3.6) is equivalent to

$$c_0 L_0'(q,x) = (\lambda_0 + qu(x)) L_0(q,x) - \lambda_0 L_1(q,x). \tag{3.13}$$

One can obtain the same differential equation from the integral equation (3.12), which corresponds to $c_0 \geq 0$.

The equation

$$c_1 L_1'(q,x) = -\lambda_1 L_0(q,x) + (\lambda_1 + qu(x)) L_1(q,x) \tag{3.14}$$

is derived from (3.11) in the same way. Cf (3.13)-(3.14) with equations [23, (3.10)].

Equations (3.13)-(3.14) can be rewritten in the matrix form:

$$\mathscr{C} \mathbf{L}'(q,x) = [-\Lambda + qu(x) \mathrm{I}] \mathbf{L}(q,x), \qquad x > 0. \tag{3.15}$$

The solution to (3.15) has the form (3.9).

Let's continue with the boundary values.

If both velocities are negative, $0 > c_0 > c_1$, then as $x \downarrow 0$,

$$\Xi_0(x),\ \Xi_1(x) \to 0, \qquad a.s. \tag{3.16}$$

This gives the trivial boundary condition: $L_0(q,x)|_{x\downarrow 0} = L_1(q,x)|_{x\downarrow 0} = 1$.

In the case of velocities with opposite signs, $c_0 > 0 > c_1$, due to the integral equation (3.12) and by (3.5) at $x = 0$, we get

$$\begin{aligned} \ell_q^{(0)} &= \int_0^\infty \lambda_0 e^{-\lambda_0 \tau} \cdot \exp(-q\phi_0(0,\tau)) L_1(q, c_0\tau) d\tau \\ &= \int_0^\infty \lambda_0 e^{-(\lambda_0 + q\alpha)\tau} \cdot \exp\left(-qc_0\beta\tau^2/2\right) L_1(q, c_0\tau) d\tau. \end{aligned} \tag{3.17}$$

This boundary condition will be stated explicitly later. However, due to (3.11), the boundary condition for L_1 at $x = 0$ is still trivial: $\ell_q^{(1)} = 1$. □

Corollary 3.2. *The coupled differential equations* (*given in matrix form by* (3.15)) *can be converted into two second-order equations separate for L_0 and L_1*:

$$\begin{cases} \dfrac{\partial^2 L_0}{\partial x^2} = \left[\dfrac{\lambda_0 + qu(x)}{c_0} + \dfrac{\lambda_1 + qu(x)}{c_1}\right]\dfrac{\partial L_0}{\partial x} + \dfrac{q}{c_0}\left[u'(x) - \dfrac{u(x)(qu(x)+2\lambda)}{c_1}\right]L_0, \\ \dfrac{\partial^2 L_1}{\partial x^2} = \left[\dfrac{\lambda_0 + qu(x)}{c_0} + \dfrac{\lambda_1 + qu(x)}{c_1}\right]\dfrac{\partial L_1}{\partial x} + \dfrac{q}{c_1}\left[u'(x) - \dfrac{u(x)(qu(x)+2\lambda)}{c_0}\right]L_1. \end{cases} \tag{3.18}$$

Proof. First, from (3.15) we get

$$\begin{aligned} L_1 =& \lambda_0^{-1}\left[(\lambda_0 + qu(x))L_0 - c_0 L_0'\right], \\ L_0 =& \lambda_1^{-1}\left[(\lambda_1 + qu(x))L_1 - c_1 L_1'\right]. \end{aligned}$$

Then, using a technique similar to Kac's trick, [12], we obtain (3.18). ☐

We provide more details in the next section.

4. Laplace transform: details

Let $z_1 = z_1(q,x)$ and $z_2 = z_2(q,x)$ be the eigenvalues of the matrix $\mathscr{M}(q,x)$, (3.7), i.e. roots of the equation

$$\det(\mathscr{M}(M(q,x) - z\mathrm{I}) = 0.$$

The latter equation has the form

$$z^2 - \mathrm{Tr}[\mathscr{M}(q,x)]\cdot z + \det[\mathscr{M}(q,x)] = 0, \tag{4.1}$$

where

$$\mathrm{Tr}[\mathscr{M}(q,x)] = \frac{\lambda_0 + qu(x/2)}{c_0} + \frac{\lambda_1 + qu(x/2)}{c_1}, \tag{4.2}$$

and

$$\begin{aligned} \det[\mathscr{M}(q,x)] =& \left(\frac{\lambda_0 + qu(x/2)}{c_0}\right)\cdot\left(\frac{\lambda_1 + qu(x/2)}{c_1}\right) - \frac{\lambda_0}{c_0}\cdot\frac{\lambda_1}{c_1} \\ =& \frac{q^2 u(x/2)^2 + 2\lambda qu(x/2)}{c_0 c_1}. \end{aligned} \tag{4.3}$$

Therefore,

$$z_1 = \nu_+(q,x) - D(q,x), \qquad z_2 = \nu_+(q,x) + D(q,x), \tag{4.4}$$

where

$$D = D(q,x) = \sqrt{\nu_+(q,x)^2 - \det[\mathscr{M}(q,x)]} = \sqrt{\nu_-(q,x)^2 + \mu_0\mu_1}, \tag{4.5}$$

$$\mu_0 = \frac{\lambda_0}{c_0}, \qquad \mu_1 = \frac{\lambda_1}{c_1},$$

and

$$\nu_\pm = \nu_\pm(q,x) = \frac{1}{2}\frac{\lambda_0 + qu(x/2)}{c_0} \pm \frac{1}{2}\frac{\lambda_1 + qu(x/2)}{c_1}. \tag{4.6}$$

In the case of both negative velocities, $0 > c_0 > c_1$, both eigenvalues z_1, z_2 are real negative, whereas if $c_0 > 0 > c_1$, then $z_1 < 0 < z_2$.

The form of the Laplace transform $\mathbf{L}(q,x)$ depends on the signs of the parameters.

Lemma 4.1.

$$\exp(x\mathscr{M}(q,x))=\begin{pmatrix} g_0+g_1\cdot\dfrac{qu(x/2)+\lambda_0}{c_0} & g_1\cdot\dfrac{qu(x/2)-\lambda_0}{c_0} \\ \\ g_1\cdot\dfrac{qu(x/2)-\lambda_1}{c_1} & g_0+g_1\cdot\dfrac{qu(x/2)+\lambda_1}{c_1}\end{pmatrix}, \tag{4.7}$$

where

$$g_0=g_0(q,x)=\frac{z_2\mathrm{e}^{xz_1}-z_1\mathrm{e}^{xz_2}}{z_2-z_1},\qquad g_1=g_1(q,x)=\frac{\mathrm{e}^{xz_2}-\mathrm{e}^{xz_1}}{z_2-z_1}. \tag{4.8}$$

Proof. By the Cayley-Hamilton Theorem

$$\exp(x\mathscr{M}(q,x))=g_0(q,x)\cdot\mathrm{I}+g_1(q,x)\cdot\mathscr{M}(q,x), \tag{4.9}$$

where $g_0(q,x)$ and $g_1(q,x)$ are defined by (4.8), see [7].

Formula (4.7) follows from (4.9) and (3.7). □

Theorem 4.2. *Laplace transform* $\mathbf{L}(q,x)$ *of the r.v.* $\Xi(x)$ *has an explicit form.*

- *Let* $0>c_0>c_1$. *Therefore,*

$$\mathbf{L}(q,x)=\exp\left(x\nu_+(q,x)\right)\left[\left(\cosh(xD(q,x))-\nu_+(q,x)\frac{\sinh(xD(q,x))}{D(q,x)}\right)\mathbf{1}\right.\\ \left.+q\frac{\sinh(xD(q,x))}{D(q,x)}\boldsymbol{w}\right], \tag{4.10}$$

where $\boldsymbol{w}=\mathscr{C}^{-1}\mathbf{1}=(1/c_0,\ 1/c_1)^{\mathrm{T}}$.

- *Let* $c_0>0>c_1$. *Therefore,*

$$L_0(q,x)=\mathrm{e}^{xz(q,x)}\ell_q^{(0)},\qquad L_1(q,x)=\mathrm{e}^{xz(q,x)}, \tag{4.11}$$

where $z(q,x)=\nu_+-D$ *is a unique negative eigenvalue of the matrix* $A(q,x)$ *and*

$$\ell_q^{(0)}=\int_0^\infty\lambda_0\mathrm{e}^{-(\lambda_0+q\alpha-c_0z(q,c_0\tau))\tau}\mathrm{e}^{-c_0q\beta\tau^2/2}\mathrm{d}\tau. \tag{4.12}$$

Proof. **a) Negative velocities, $\mathbf{0>c_0>c_1}$.**

Since in this case, $\det[\mathscr{M}(q,x)]>0$, and $\nu_+<0$, (see (4.3), (4.6), (4.2)) then equation (4.1) has two real negative roots z_1,z_2, (4.4).

Due to the Cayley-Hamilton Theorem, (4.9), the Laplace transform $\mathbf{L}(q,x)$ is given by

$$\mathbf{L}(q,x)=[g_0(q,x)\cdot\mathrm{I}+g_1(q,x)\cdot\mathscr{M}(q,x)]\,\mathbf{1},$$

Substituting $z_1=\nu_+(q,x)-D(q,x)$, $z_2=\nu_+(q,x)+D(q,x)$ into (4.8) and (4.9), we obtain: for $q>0$,

$$g_0=\mathrm{e}^{x\nu_+(q,x)}\left[\cosh(xD(q,x))-\nu_+(q,x)\frac{\sinh(xD(q,x))}{D(q,x)}\right],\quad g_1=\mathrm{e}^{x\nu_+(q,x)}\frac{\sinh(xD(q,x))}{D(q,x)}. \tag{4.13}$$

Since $\Lambda\mathbf{1}=\mathbf{0}$, then by (3.7)

$$\mathbf{L}(q,x)=g_0(q,x)\mathbf{1}+qg_1(q,x)\boldsymbol{w}. \tag{4.14}$$

From (4.13) and (4.14), we obtain (4.10).

Condition (3.8) is satisfied since both eigenvalues are negative, $z_1<z_2<0$.

b) Velocities with opposite signs $\mathbf{c_0>0>c_1}$.

Note that by the Cayley-Hamilton theorem, $\mathbf{L}(q,x)$ is formed as a linear combination of two exponentials $\mathrm{e}^{xz_1(q,x)}$ and $\mathrm{e}^{xz_2(q,x)}$ with algebraic coefficients. Since in the case of velocities of opposite signs $\det[A(q,x)]<0$, see (4.3), then in this case only one eigenvalue is negative. In view of the zero condition at infinity (3.8), the Laplace transform L has the form $\mathbf{L}(q,x)=\mathrm{e}^{xz(q,x)}\boldsymbol{\ell}_q$, where $z(q,x)=\nu_+(q,x)-D(q,x)$ is the only negative eigenvalue,

Setting $x=0$ in (3.11), we find that $\ell_q^{(1)}=1$, – and therefore we obtain (4.11) . Then, according to (3.17) and (4.11), we obtain (4.12). □

5. Moments

Notice that the n-th moment $\mathbf{M}_n(x)=(M_n^{(0)}(x),M_n^{(1)}(x))$ of $\Xi(x)$, when it exists, is determined by the derivative,

$$\begin{aligned} M_n^{(0)}(x)=&\mathbb{E}_0\left[\Xi(x)^n\right]=(-1)^n\frac{\partial^n}{\partial q^n}[L_0(q,x)]|_{q=0},\\ M_n^{(1)}(x)=&\mathbb{E}_1\left[\Xi(x)^n\right]=(-1)^n\frac{\partial^n}{\partial q^n}[L_1(q,x)]|_{q=0},\end{aligned}\qquad n\geq 1.$$

With this in hand, by virtue of (3.15), we obtain the following set of first-order equations:

$$\mathscr{C}\mathbf{M}_n'(x)=-\Lambda\mathbf{M}_n(x)-nu(x)\mathbf{M}_{n-1}(x),\qquad x>0,\quad n\geq 1,\tag{5.1}$$

with $\mathbf{M}_0(x)\equiv\mathbf{1}$. Therefore, the moments satisfy the recurrence relation

$$\mathbf{M}_n(x)=\boldsymbol{m}_n(x)-n\int_0^x u(y)\mathscr{E}(x-y)\mathscr{C}^{-1}\mathbf{M}_{n-1}(y)\mathrm{d}y,\qquad x>0,\quad n\geq 1.\tag{5.2}$$

Here $\boldsymbol{m}_n(x)=(m_n^{(0)}(x),\,m_n^{(1)}(x))^{\mathrm{T}}=\mathscr{E}(x)\mathbf{M}_n(0)$, and $\mathscr{E}(x)$ is the matrix exponential with $\mathscr{C}^{-1}\Lambda=\begin{pmatrix}-\mu_0 & \mu_0\\ \mu_1 & -\mu_1\end{pmatrix}$, $\mathscr{E}(x)=\exp(-x\mathscr{C}^{-1}\Lambda)$, where, recall, $\mu_0=\lambda_0/c_0$, $\mu_1=\lambda_1/c_1$.

The matrix exponential $\mathscr{E}(x)$ can be represented explicitly using the Cayley-Hamilton theorem.

Lemma 5.1.
- *If $2\mu:=\mu_0+\mu_1\neq 0$, then*

$$\mathscr{E}(x)=\mathscr{E}_\mu(x)=\exp\left(-xC^{-1}\Lambda\right)=\frac{1}{2\mu}\begin{pmatrix}\mu_1+\mu_0\mathrm{e}^{2\mu x} & \mu_0(1-\mathrm{e}^{2\mu x})\\ \mu_1(1-\mathrm{e}^{2\mu x}) & \mu_0+\mu_1\mathrm{e}^{2\mu x}\end{pmatrix};\tag{5.3}$$

- *if $\mu_0+\mu_1=0$, then*

$$\mathscr{E}_0(x)=\mathrm{I}-x\mathscr{C}^{-1}\Lambda=\begin{pmatrix}1+\mu_0x & -\mu_0x\\ -\mu_1x & 1+\mu_1x\end{pmatrix}.\tag{5.4}$$

Proof. See Lemma 4.1 with $q=0$. □

Equations (5.1)-(5.2) are provided with the boundary value $\mathbf{M}_n(0)$. Due to (3.16) and (3.17), $\mathbf{M}_n(0)$ has the form

$$\mathbf{M}_n(0)=\lim_{x\downarrow 0}\mathbf{M}_n(x)=(\xi_n,\,0)^{\mathrm{T}},\tag{5.5}$$

where

$$\xi_n = \begin{cases} 0, & \text{if } 0 > c_0 > c_1, \\ (-1)^n \lim_{x\downarrow 0} \dfrac{\mathrm{d}^n}{\mathrm{d}q^n} L_0(q,x)|_{q=0}, & \text{if } c_0 > 0 > c_1, \end{cases} \qquad n \geq 1. \tag{5.6}$$

Let us clarify the details for $n = 1$. According to (5.2), the mean value $\mathbf{M}_1(x) = (M_1^{(0)}(x),\, M_1^{(1)}(x))^{\mathrm{T}}$ of $\Xi(x)$ has the form of a sum of two terms,

$$\mathbf{M}_1(x) = \boldsymbol{m}_1(x) + \mathbf{A}(x), \tag{5.7}$$

where

$$\boldsymbol{m}_1(x) = \mathscr{E}(x)\mathbf{M}_1(0) = \mathscr{E}(x)(\xi_1,\, 0)^{\mathrm{T}},$$

with ξ_1 defined by (5.6), and

$$\mathbf{A}(x) = -\int_0^x u(y)\mathscr{E}(x-y)\mathbf{w}\mathrm{d}y = (A^{(0)}(x),\, A^{(1)}(x))^{\mathrm{T}}$$

with $\mathbf{w} = \mathscr{C}^{-1}\mathbf{1} = (1/c_0,\, 1/c_1)^{\mathrm{T}}$.

Lemma 5.2. *The first term of* (5.7) *is given by*

$$\boldsymbol{m}_1(x) = \xi_1 \cdot \begin{cases} \dfrac{1}{\mu_0+\mu_1}\left(\mu_1 + \mu_0 \mathrm{e}^{2\mu x},\, \mu_1(1-\mathrm{e}^{2\mu x})\right)^{\mathrm{T}}, & \text{if } \mu_0+\mu_1 \neq 0\,, \\[2ex] (1+\mu_0 x,\, -\mu_1 x)^{\mathrm{T}}, & \text{if } \mu_0+\mu_1 = 0\,. \end{cases} \tag{5.8}$$

The second term is

$$\mathbf{A}(x) = \begin{cases} -\dfrac{\lambda_0+\lambda_1}{\lambda_0 c_1 + \lambda_1 c_0} \cdot x u(x/2)\mathbf{1} + \dfrac{c_0 - c_1}{\lambda_0 c_1 + \lambda_1 c_0}\Psi_\mu(x)\mathbf{e}, & \text{if } \mu_0+\mu_1 \neq 0, \\[2ex] -x u(x/2)\mathbf{w} + \dfrac{c_0-c_1}{c_0 c_1} \cdot \dfrac{x^2}{2} u(x/3)\mathbf{e}, & \text{if } \mu_0+\mu_1 = 0. \end{cases} \tag{5.9}$$

Here

$$\Psi_\mu(x) = \alpha \frac{\mathrm{e}^{2\mu x}-1}{2\mu} + \beta \frac{\mathrm{e}^{2\mu x} - 1 - 2\mu x}{4\mu^2},$$

and $\mathbf{1} = (1,1)^{\mathrm{T}}$, $\mathbf{e} = (\mu_0,\, -\mu_1)^{\mathrm{T}}$.

Proof. Formula (5.8) follows from (5.5) and (5.3)-(5.4).

To make clear (5.9), notice that

$$\begin{gathered} \int_0^x u(y)\mathrm{d}y = x u(x/2), \qquad \int_0^x (x-y)u(y)\mathrm{d}y = \frac{x^2}{2}u(x/3) \\ \text{and} \\ \int_0^x \mathrm{e}^{2\mu(x-y)} u(y)\mathrm{d}y = \alpha \frac{\mathrm{e}^{2\mu x}-1}{2\mu} + \beta \frac{\mathrm{e}^{2\mu x}-1-2\mu x}{4\mu^2} = \Psi_\mu(x), \qquad \mu \neq 0. \end{gathered} \tag{5.10}$$

Hence, formulae (5.9) follow by definition, $\mathbf{A}(x) = -\int_0^x u(y)\mathscr{E}(x-y)\mathbf{w}\mathrm{d}x$, from (5.3)-(5.4), (5.8) and (5.10). □

Proposition 5.3. *Let* $\mu_0+\mu_1\neq 0$.

The first moment $\mathbf{M}_1(x)$ *is determined by* (5.7)-(5.9) *with* ξ_1 *given by*

$$\xi_1=\frac{1}{-\lambda_0^2\mu_1}\left[\alpha\mu_0+\beta+\mu_0\int_0^\infty \mathrm{e}^{-\mu_0 x}A^{(1)}(x)\mathrm{d}x\right], \tag{5.11}$$

where $A^{(1)}(x)$ *is the second entry of the vector* $\mathbf{A}(x)=(A^{(0)}(x),\,A^{(1)}(x))^{\mathsf{T}}$, *which is defined by* (5.9).

Proof. By virtue of (3.12),

$$M_1^{(0)}(x)=\int_0^\infty \lambda_0\mathrm{e}^{-\lambda_0\tau}\left[\phi_0(\tau,x)+M_1^{(1)}(x+c_0\tau)\right]\mathrm{d}\tau. \tag{5.12}$$

To determine $\xi_1=M_1^{(0)}(0)$, note that by (3.5) and (5.7), after setting $x=0$ in (5.12), we get

$$\xi_1=\int_0^\infty \lambda_0\mathrm{e}^{-\lambda_0\tau}\left[\alpha\tau+\frac{1}{2}\beta c_0\tau^2+\frac{\xi_1\mu_1}{\mu_0+\mu_1}(1-\mathrm{e}^{2\mu c_0\tau})+A^{(1)}(c_0\tau)\right]\mathrm{d}\tau.$$

Formula (5.11) follows after using (5.10). □

To make (5.11) explicit, it is sufficient to evaluate the integral $\int_0^\infty \mathrm{e}^{-\mu_0 x}A^{(1)}(x)\mathrm{d}x$, but the result looks cumbersome. In the homogeneous case $\lambda_0=\lambda_1=\lambda$ for $c_0=a+c>0>c_1=a-c$, the search for the first moment $\mathbf{M}_1(x)$ is simplified.

Corollary 5.4. *Let* $\lambda_0=\lambda_1=\lambda,\ c_0=a+c>0, c_1=a-c<0,$ *and* $x>0$.

Then,

- *for* $a>0$, *the random variable* $\Xi(x)$ *is infinite with positive probability*, *and*

$$L_0(q,x)|_{q\downarrow 0}=\mathbb{P}_0\{\Xi(x)<\infty\}=\frac{c-a}{c+a}\exp\left(-\frac{2a\lambda x}{c^2-a^2}\right), \tag{5.13}$$

$$L_1(q,x)|_{q\downarrow 0}=\mathbb{P}_1\{\Xi(x)<\infty\}=\exp\left(-\frac{2a\lambda x}{c^2-a^2}\right); \tag{5.14}$$

- *for* $a=0$,

$$\begin{gathered}\mathbb{P}_0\{\Xi(x)<\infty\}=1,\qquad \mathbb{P}_1\{\Xi(x)<\infty\}=1,\\ \textit{and}\\ \mathbb{E}_0[\Xi(x)]=+\infty,\qquad \mathbb{E}_1[\Xi(x)]=+\infty;\end{gathered} \tag{5.15}$$

- *if* $a<0$, *then*

$$\mathbb{E}_0[\Xi(x)]=\frac{c}{-a\lambda}\left[\alpha+\frac{\beta(a+c)}{\lambda}\right]+\frac{xu(x/2)}{-a}\exp\left(-\frac{2a\lambda x}{c^2-a^2}\right), \tag{5.16}$$

$$\mathbb{E}_1[\Xi(x)]=\frac{xu(x/2)}{-a}\exp\left(-\frac{2a\lambda x}{c^2-a^2}\right). \tag{5.17}$$

Proof. Since in the homogeneous case, by virtue of (4.6) and (4.3)

$$\nu_+=\frac{-a(\lambda+qu(x/2))}{c^2-a^2},\qquad \nu_-=\frac{c(\lambda+qu(x/2))}{c^2-a^2},$$

$$\det[\mathscr{M}(q,x)]=-\frac{qu(x/2)(2\lambda+qu(x/2))}{c^2-a^2},$$

then the only negative eigenvalue is

$$z(q,x) = -\frac{a(\lambda + qu(x/2)) + \sqrt{c^2 qu(x/2)(2\lambda + qu(x/2)) + \lambda^2 a^2}}{c^2 - a^2}, \qquad q \geq 0, \quad (5.18)$$

and for $c > a > 0$

$$z(q,x)|_{q\downarrow 0} = -\frac{2a\lambda}{c^2 - a^2} < 0. \quad (5.19)$$

Note that from the boundary condition at infinity (3.8) and $L_1(q,x)|_{x\downarrow 0} = 1$, we have $L_1(q,x) = \mathrm{e}^{xz(q,x)}$, which simplify equation (3.17) to

$$\ell_q^{(0)} = \int_0^\infty \lambda \exp\big(-(\lambda + q\alpha)\tau - q(a+c)\beta\tau^2/2 + (a+c)\tau z(q,(a+c)\tau)\big)\mathrm{d}\tau, \quad q > 0. \quad (5.20)$$

Hence, by (5.19),

$$\ell_q^{(0)}\Big|_{q\downarrow 0} = \int_0^\infty \lambda \exp\left(-\lambda\tau + (a+c)\frac{-2a\lambda}{c^2-a^2}\tau\right)\mathrm{d}\tau = \frac{c-a}{c+a} < 1,$$

and

$$L_1(q,x)|_{q\downarrow 0} = \exp\left(-\frac{2a\lambda x}{c^2-a^2}\right),$$

which gives (5.13)-(5.14).

From (5.18) we also get

$$\frac{\partial z(q,x)}{\partial q} = \frac{-u(x/2)}{c^2-a^2}\left(a + \frac{c^2(qu(x/2)+\lambda)}{\sqrt{c^2 qu(x/2)(2\lambda + qu(x/2)) + \lambda^2 a^2}}\right). \quad (5.21)$$

Further, if $a = 0$, then $z(q,x)|_{q=0} = 0$, by (5.20), $\ell_q^{(0)}\Big|_{q=0} = \int_0^\infty \lambda \exp(-\lambda\tau)\mathrm{d}\tau = 1$, by (5.21),

$$\frac{\partial z(q,x)}{\partial q}\bigg|_{q=0} = -\frac{u(x/2)}{c^2} \cdot \frac{c(qu(x/2)+\lambda)}{\sqrt{c^2 qu(x/2)(2\lambda + qu(x/2))}}\bigg|_{q\downarrow 0} = -\infty,$$

and, similarly,

$$\frac{\mathrm{d}\ell_q^{(0)}}{\mathrm{d}q}\bigg|_{q=0} = -\infty.$$

Hence, (5.15) is satisfied by virtue of (4.11).

If $a < 0$, then by (5.18) and (5.21)

$$z(q,x)|_{q=0} = 0, \qquad \frac{\partial z(q,x)}{\partial q}\bigg|_{q=0} = \frac{-u(x/2)}{c^2-a^2}\left(a - \frac{c^2\lambda}{a\lambda}\right) = \frac{u(x/2)}{a}.$$

Therefore, by virtue of (3.17), (3.5) and (5.20),

$$\ell_q^{(0)}\Big|_{q=0} = \int_0^\infty \lambda \mathrm{e}^{-\lambda\tau}\mathrm{d}\tau = 1,$$

$$\begin{aligned}\frac{\mathrm{d}\ell_q^{(0)}}{\mathrm{d}q}\bigg|_{q=0} &= -\int_0^\infty \lambda\left(\alpha\tau + (a+c)\beta\tau^2/2 - (a+c)\tau\frac{\partial z(q,(a+c)\tau)}{\partial q}\right)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau \\ &= -\int_0^\infty \lambda\Big(\alpha\tau + (a+c)\beta\tau^2/2 - (a+c)\tau u((a+c)\tau/2)/a\Big)\mathrm{e}^{-\lambda\tau}\mathrm{d}\tau.\end{aligned}$$

The latter integral can be written explicitly:

$$\left.\frac{\mathrm{d}\ell_q^{(0)}}{\mathrm{d}q}\right|_{q=0} = -\frac{\alpha}{\lambda} - \frac{(a+c)\beta}{\lambda^2} + \frac{(a+c)\alpha}{a\lambda} + \beta\frac{(a+c)^2}{a\lambda^2} = \frac{c}{a\lambda}\left[\alpha + \frac{\beta(a+c)}{\lambda}\right]. \quad (5.22)$$

Differentiating with respect to q in (4.11), we obtain (5.16)-(5.17). □

6. Kac's scaling

The distribution of the first passage area $\mathscr{A}(x)$ could be analysed in more detail and compared with the case of Brownian motion.

Let $u(x) = x$, i.e. $\alpha = 0$, $\beta = 1$.

Notice that in the case of a symmetric inhomogeneous telegraph process, $c_0 = -c_1 = c > 0$, system (3.18) takes the form:

$$\begin{cases} \dfrac{\partial^2 L_0}{\partial x^2} = \left[\dfrac{\lambda_0 - \lambda_1}{c}\right]\dfrac{\partial L_0}{\partial x} + \left[\dfrac{q}{c} + \dfrac{qx(qx+2\lambda)}{c^2}\right]L_0, \\ \dfrac{\partial^2 L_1}{\partial x^2} = \left[\dfrac{\lambda_0 - \lambda_1}{c}\right]\dfrac{\partial L_1}{\partial x} + \left[-\dfrac{q}{c} + \dfrac{qx(qx+2\lambda)}{c^2}\right]L_1. \end{cases}$$

Further, in the case of symmetric switching, $\lambda_0 = \lambda_1 = \lambda$, and asymmetric velocities, $c_0 = a + c$, $c_1 = a - c$, system (3.18) is simplified to

$$\begin{cases} (a^2 - c^2)\dfrac{\partial^2 L_0}{\partial x^2} = 2a(\lambda + qx)\dfrac{\partial L_0}{\partial x} + [q(a - c - 2\lambda x) - q^2x^2]L_0, \\ (a^2 - c^2)\dfrac{\partial^2 L_1}{\partial x^2} = 2a(\lambda + qx)\dfrac{\partial L_1}{\partial x} + [q(a + c - 2\lambda x) - q^2x^2]L_1; \end{cases} \quad (6.1)$$

Under Kac's scaling, i.e. $c \to \infty$, $\lambda_0 = \lambda_1 = \lambda \to \infty$ such that $c^2/\lambda \to 1$, the process X weakly converges to the Brownian motion with drift, [29, Sect.2.7]. On the other hand, both equations of system (6.1) become

$$\frac{1}{2}\frac{\partial^2 L}{\partial x^2} = -a\frac{\partial L}{\partial x} + qxL, \quad (6.2)$$

with the boundary condition $\mathbf{L}(0,q) = 1$ which coincides with the equation for the Laplace transform [2, (3.10)] for the Brownian motion with drift.

In the completely symmetric telegraph process, i.e. if $\lambda_0 = \lambda_1 = \lambda$, $a = 0$, equation (6.2) becomes Airy equation, and the Laplace transform of the first passage area $\mathscr{A}(x)$ is given through the Airy function $\mathrm{Ai}(x)$:

$$\mathbb{E}\mathrm{e}^{-q\mathscr{A}(x)} = L(q,x) = 3^{2/3}\Gamma\left(\frac{2}{3}\right)\mathrm{Ai}(2^{1/3}q^{1/3}x).$$

See [13, 14, 2].

The inverse Laplace transform leads to the first passage area density function:

$$f(y,x) = \frac{2^{1/3}}{3^{2/3}\Gamma(1/3)} \cdot \frac{x}{y^{4/3}} \exp\left(-2x^3/9y\right).$$

Under Kac's scaling, $\lambda, c \to +\infty$, $c^2/\lambda \to 1$, the limit of $\mathbf{M}_1(x)$ coincide with known result for the Brownian motion (with drift).

Theorem 6.1. *Let* $c_0 = a + c,\ c_1 = a - c,\ c \to \infty$, *and* $\lambda_0 = \lambda_1 = \lambda \to \infty$ *such that*

$$c^2/\lambda \to 1.$$

Therefore,

- *if* $a = 0$, *then* $\mathbf{M}_1(x) \to \infty$;
- *if* $a < 0$, *then*

$$\mathbf{M}_1(x) \to -\left[\alpha\frac{x}{a} + \beta\left(\frac{x^2}{2a} - \frac{x}{2a^2}\right)\right]\mathbf{1}. \tag{6.3}$$

Proof. If $a = 0$ then this is a completely symmetric case with velocities of opposite signs. In this case, by virtue of (5.18) and (5.20), the unique negative eigenvalue is given by

$$z(q,x) = -\frac{1}{c}\sqrt{qu(x/2)(qu(x/2) + 2\lambda)},$$

and

$$\ell_q^{(0)} = \frac{\lambda}{\lambda + q\alpha - \frac{c+a}{c}\sqrt{q(q+2\lambda)}}.$$

Therefore

$$\frac{\mathrm{d}\ell_q^{(0)}}{\mathrm{d}q}|_{q=0} = -\infty,$$

which by virtue of (5.8), leads to

$$m_0(x),\ m_1(x) \to \infty.$$

Meanwhile $\mathbf{A}(x)$ remain finite.

In the case $a < 0$ and $\alpha = 1, \beta = 0$ (the expectation of $\mathscr{T}(x)$), by (5.22) and (5.18) we have

$$-\frac{\mathrm{d}\ell_q^{(0)}}{\mathrm{d}q}|_{q=0} = \frac{c}{\lambda a},$$

and therefore,

$$m_0(x),\ m_1(x) \to \frac{1 - \mathrm{e}^{-2ax}}{2a^2}.$$

Note that by (5.10)

$$\mu_0\frac{c_0 - c_1}{\lambda_0 c_1 + \lambda_1 c_0}\Psi_\mu(x) \quad \text{and} \quad \mu_1\frac{c_0 - c_1}{\lambda_0 c_1 + \lambda_1 c_0}\Psi_\mu(x) \to -\frac{1 - \mathrm{e}^{-2ax}}{2a^2}.$$

By virtue of (5.7), we obtain (6.3). The case $\alpha = 0,\ \beta = 1$ is treated similarly. □

Remark 6.2. The limiting behaviour of $\mathbf{M}_1(x)$ under Kac's scaling, found in Theorem 6.1, isconsistent with the moments of $\Xi(x)$ in the case of Brownian motion with drift, see [2, Example 1].

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