Global and Stochastic Analysis
Vol. 11 No. 4 (September, 2024)

Received: 30th June 2024 Revised: 29th July 2024 Accepted: 24th August 2024

NUMERICAL ALGORITHM AND COMPUTATIONAL EXPERIMENTS FOR A CERTAIN LINEAR STOCHASTIC OSKOLKOV MODEL

E. A. SOLDATOVA* AND A. V. KELLER

Abstract. Investigated is a model of dynamics of pressure and velocity of a viscoelastic incompressible fluid moving with the random external effect; it is based on stochastic Oskolkov equations with the Showalter – Sidorov initial condition. The article describes an algorithm for a numerical solution of the Showalter – Sidorov problem for stochastic Oskolkov equations; the algorithm is based on the Galerkin method. Provided is a numerical investigation algorithm providing for numerical solutions for both degenerate and non-degenerate equations. The main theoretical results that enabled this numerical investigation are the methods of the theory of degenerate groups of operators and of the theory of the Sobolev type equations. The algorithms are represented by schemes enabling building flowcharts of programs for computational experiments. Results of computational experiments. In addition, numerical investigation of the stochastic model involves further obtaining and processing the results of n experiments at various values of a random variable, including those related to rare events.

1. Introduction

Modern analytical and numerical investigations of a number of non-classical mathematical models are closely related to the expansion of the applicability of current methods for assessing the states and parameters of complex systems when it comes to solving problems pertaining to fluid motion, improvement of efficiency of oil production, optimization of automobile traffic and many other problems. Scientists and researchers have been using for several decades Sobolev type stochastic equations to describe and model a large number of physical, technical and technological processes

$$Ld\zeta = M\zeta dt + NdW. \tag{1.1}$$

It should be noted that Sobolev-type stochastic equations in relation to dynamical systems are based on differential equations that take into account a large number of perturbations acting on an object. One example of such a process is the motion of a particle in a liquid under the action of chaotic collisions of molecules.

Key words and phrases. Oskolkov model, geometric graph, Showalter – Sidorov initial condition, numerical investigation, algorithm, Sobolev type stochastic equations, computational experiment.

* Corresponding author.

Such a motion was discovered by R. Brown in the 19th century, and is called the Brownian motion [1]. The apparatus describing the process of changing random variables in development is the theory of random functions (random processes). The theory of random processes is widely used in the analytical and numerical investigation of various dynamic mathematical models. This is due to the fact that the operation of the studied systems is affected by external and internal interference, external influences or, in other words, so-called "noises".

Traditionally, the first use of stochastic differential equations in the literature is associated with the work on the description of Brownian motion, made independently by M. von Smolukhovsky in 1904 and A. Einstein in 1905.

M. Smolukhovsky, using the kinetic law of energy distribution, created the theory of Brownian motion, which proved the validity of the kinetic theory of heat and contributed to its final approval [10]. A. Einstein, inspired by the work of M. Smolukhovsky, published an essential work on the theory of Brownian motion [3].

However, stochastic differential equations were used a little earlier in 1900 by the French mathematician L. Bachelier in his doctoral dissertation "Theory of speculation" [2]. Using on the ideas of the above work, the French physicist P. Langevin began to apply stochastic differential equations in works on physics. Later, he and the Russian physicist R. Stratonovich developed a more rigorous mathematical justification for stochastic differential equations.

The Sobolev type stochastic equations in the sense of the Nelson – Gliklich derivative were first investigated in [4]–[7], [12]. The historiography of Sobolev type stochastic equations, as well as the two main approaches to the investigation of equations of this type, have been described in more detail in [11]. Therefore, new results for the theory of Sobolev type stochastic equations enabling the investigation of various mathematical models with the development of numerical methods and algorithms are relevant.

Let us turn to linear mathematical Oskolkov model [9]. Of note, in the framework of this investigation, the Oskolkov equations

$$\lambda z_{jt} - z_{jxxt} = \alpha z_{jxx} + f_j \tag{1.2}$$

are considered on a finite connected oriented graph.

Suppose that $\boldsymbol{G} = \boldsymbol{G}(\mathfrak{V};\mathfrak{E})$ is a geometric graph, $\mathfrak{V} = \{V_i\}$ denotes a set of vertices and $\mathfrak{E} = \{E_j\}$ denotes a set of edges. On the edges $\mathfrak{E}_j$ of the graph $\boldsymbol{G}$ let us consider the linear stochastic Oskolkov equations

$$\lambda d\zeta_j - d\zeta_{jxx} = \beta_j \zeta_{jxx} dt + N dW_j, \tag{1.3}$$

with Showalter – Sidorov initial condition

$$P(\zeta(0) - \xi_0) = 0, \tag{1.4}$$

where $W = (W_1, W_2, ..., W_n)$ is an $\mathfrak{F}$-digit nuclear K-Wiener process, operator $K \in \mathcal{L}(\mathfrak{Z})$ is nuclear, P_0 is a relatively spectral projector. At the vertices V_i of the graph $\boldsymbol{G}$ let us set continuity conditions

$$\begin{gathered}\zeta_j(0,t) = \zeta_k(0,t) = \zeta_m(l_m,t) = \zeta_n(l_n,t),\\ E_j, E_k \in E^{\alpha}(V_i), E_m, E_n \in E^{\omega}(V_i)\end{gathered} \tag{1.5}$$

and flow balance conditions

$$\sum_{E_j\in E^{\alpha}(V_i)} d_j\zeta_{jx}(0,t) - \sum_{E_k\in E^{\omega}(V_i)} d_k\zeta_{kx}(l_k,t) = 0, \tag{1.6}$$

where $E^{\alpha}(V_i)$ ($E^{\omega}(V_i)$) is the set of edges having, at the vertices V_i which is the beginning (end), $l_j > 0$ and $d_j > 0$ which are the length of the edge and the diameter of its cross section. Equations (1.3) simulate the dynamics of pressure and velocity of a viscoelastic incompressible fluid moving in the jth section of the pipeline. An example of such a liquid may be high-paraffin grades of oil which are produced in the fields of Western Siberia. The $\beta \in \mathbb{R}\setminus\{0\}$ parameter characterizes the elasticity of the liquid, the $\lambda \in \mathbb{R}$ parameter describes the viscosity of the liquid, the random $\zeta_j = \zeta_j(x,t), (x,t) \in (0,l_j)\times\mathbb{R}$ process characterizes the change in velocity and pressure of a viscoelastic incompressible liquid in the jth section of the pipeline.

On the other hand, the resulting one-dimensional analogue (1.3) of the system (1.2) can be interpreted as a change in the speed of the traffic flow, identifying in this case the flow of vehicles with the hydrodynamic flow [14].

The Oskolkov equations on a graph were first investigated by G.A. Sviridyuk, A.S. Shipilov [13], S.A. Zagrebina, E.A. Soldatova [15]. In addition, the monograph by T.G. Sukacheva and O.P. Matveeva considers mathematical models of viscoelastic incompressible liquids [8].

Of note, the linear stochastic Oskolkov model of change in the dynamics of velocity and pressure of an incompressible viscoelastic fluid with the Showalter – Sidorov initial condition in the sense of the classical direction was analytically examined in [15]. The solvability of the Showalter – Sidorov problem (1.4) for an abstract Sobolev type linear stochastic equation (1.3) has been proved.

Theorem 1.1. [15] *Suppose the process W is the $\mathfrak{F}^1$-digit K-Wiener process, the operator $N \in \mathcal{L}(\mathfrak{F}^1)$ and for each fixed t random variables $\xi_0 \in \mathbf{L_2}(\Omega, \mathfrak{Z}^1)$ and the W process are independent. Then for any $\xi_0 \in \mathbf{L_2}(\Omega, \mathfrak{Z}^1)$ the problem (1.3), (1.4) has a unique solution defined by expression*

$$\zeta(t) = Z^t\xi_0 + \textstyle\int_0^t Z^{t-s}L_1^{-1}NW(s)ds, \tag{1.7}$$

where Z_0^t

$$Z^t = \frac{1}{2\pi i}\int_{\gamma}(\mu L - M)^{-1}Le^{\mu t}d\mu.$$

2. The Algorithm of the Numerical Method

We will look for an approximate solution of the problem (1.3)–(1.6) in the form of

$$\tilde{\zeta}(x,t) = \left(\tilde{\zeta}_1(x,t), \tilde{\zeta}_2(x,t), ..., \tilde{\zeta}_n(x,t)\right), \tag{2.1}$$

where $\tilde{\zeta}_j(x,t)$ is the approximate solution on the jth beam of the graph of the form

$$\tilde{\zeta}_j(x,t) = \zeta_j^N(x,t) = \sum_{k=1}^{N} a_k(t)\varphi_k^j(x). \tag{2.2}$$

Here, $\{\varphi_k\} = \{\varphi_k^1, \varphi_k^2, ..., \varphi_k^N\}$ refer to the corresponding orthonormal eigenfunctions relative to the scalar product of $\boldsymbol{L_2(G)}$.

Next, we apply the representation (2.2) to the Oskolkov equation (1.3), resulting in a system of equations by scalarly multiplying by eigenfunctions. Each system's equation will contain only one unknown Galerkin coefficient; therefore, the expression (2.2) is a partial sum of the series, whose convergence provides for the convergence of the approximate solution to the exact one.

Of note, the linearized Oskolkov model is considered on the graph. Let us build a description of the algorithm of the numerical solution.

Step 1. Given: N is the number of summands of the Galerkin sum; l_j d_j are the length and cross-sectional area of the edges of the graph, respectively (equal for all edges), ω is the parameter of external action, a random variable A.
λ_j, β_j are the coefficients of the linear Oskilkov equation. The coefficients are taken equal for all edges of the graph.

Step 2. Showalter – Sidorov initial conditions are set. $\xi_0(x)$ is a function of the Showalter – Sidorov initial condition, whose coefficients are normally distributed random variables.

Step 3. Generation of approximate solutions

$$\tilde{\zeta}_j(t,x) = \sum_{k=1}^{N} a_k(t)\varphi_k^j(x), \tag{2.3}$$

and application of (2.1) to the equation.

Step 4. Using the differential equation from the previous step relative to the unknown variables $a_k(t)$, we will multiply it scalarly by functions $\varphi_k^j(x)$, $k = 1, ..., N$, to obtain a system of differential equations.

Step 5. Random variables ξ_{0k} are generated.

Step 6. For numbers k_0, for which the λ parameter coincides with the eigenvalue ν_{k_0} of the Δ operator, a system of corresponding algebraic equations is made and solved.

Step 7. For numbers k_0, for which the λ parameter does not coincide with the eigenvalue ν_{k_0} of the Δ operator, a system of corresponding differential equations is made and solved.

Step 8. The system of differential equations is solved with Showalter – Sidorov initial conditions.

3. Computational Experiments

Consider a five-edge graph with six vertices, shown in Figure 1, the lengths of all edges are equal: $l_j = 2\pi$ and the diameter of the sections is the same for all edges $d_j = 1$, $j = 1, 2, ..., 5$.

For such a graph, let us write down the continuity conditions (1.5)

$$\begin{gathered} \zeta_1(\pi) - \zeta_2(0) = 0, \\ \zeta_1(\pi) - \zeta_4(0) = 0, \\ \zeta_2(\pi) - \zeta_3(0) = 0, \\ \zeta_4(\pi) - \zeta_5(0) = 0 \end{gathered} \tag{3.1}$$

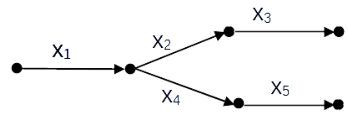

FIGURE 1. The graph for computational experiment

and the flow balance conditions (1.6)

$$\begin{gathered}\zeta_{1x}(0)=0,\\ \zeta_{1x}(2\pi)-\zeta_{2x}(0)-\zeta_{3x}(0)=0,\\ \zeta_{2x}(2\pi)-\zeta_{3x}(0)=0,\\ \zeta_{4x}(2\pi)-\zeta_{5x}(0)=0,\\ \zeta_{3x}(\pi)=0,\\ \zeta_{5x}(\pi)=0.\end{gathered}\tag{3.2}$$

With the given coefficients $\beta=0,4$, $\lambda=25$, the Oskolkov equations will be set on the edges of the graph

$$25d\zeta_j+d\zeta_{jxx}=0,4\zeta_{jxx}dt+NdW_j,\quad j=1,2,...,5.\tag{3.3}$$

We will look for solutions $\zeta_j(t,x)$, $j=1,2,...,5$ of this problem in the form of Galerkin sums, by taking 6 summands.

$$\zeta_j(t,x)=\sum_{k=1}^{6}a_k(t)\varphi_{j,k}(x),\quad j=1,2,...,5.$$

Having solved the Sturm–Liouville problem, we get

$$\begin{gathered}\lambda_k=k^2,\\ \varphi_{1,k}(x)=\sqrt{\tfrac{2}{15\pi}}cos(kx),\\ \varphi_{2,k}(x)=\sqrt{\tfrac{1}{30\pi}}(\cos(kx)+\cos(k(x-4\pi))+\cos(k(x-8\pi))),\\ \varphi_{3,k}(x)=\sqrt{\tfrac{1}{120\pi}}(\cos(k(x+6\pi))+\cos(k(x-10\pi))+2\cos(k(x-2\pi))+\\ +\cos(k(x+2\pi))+\cos(k(x-6\pi))),\\ \varphi_{4,k}(x)=\sqrt{\tfrac{1}{30\pi}}(\cos(k(x-2\pi))+\cos(k(x-6\pi))),\\ \varphi_{5,k}(x)=\sqrt{\tfrac{1}{30\pi}}(\cos(kx)+\cos(k(x-4\pi))).\end{gathered}$$

Of note, $\lambda=\lambda_5$.

We will assume that the system is exposed to the same effect, therefore all random variables here are normally distributed Gaussian quantities $\sim N(0;1,25)$.

Using the Showalter – Sidorov initial conditions, let us obtain a representation for the initial values considered as follows

$$P(\zeta(0)-\xi_0)=\sum_{k:\mu_k\in\sigma^L(M)}\langle\zeta(0,x)-\xi_0(x),\varphi_k\rangle\varphi_k=0,\tag{3.4}$$

further $\xi_0=(\xi_{1,0},\xi_{2,0},\xi_{3,0},\xi_{4,0},\xi_{5,0})$ according to the number of edges $j=5$.

As a result of the generation of random variables included in the decomposition for the functions of the initial condition, we obtained

$$\begin{aligned}
&\xi_{1,0_1} = -3,27506776941, && \xi_{1,0_2} = -1,36125225452, && \xi_{1,0_3} = -0,0374554524740,\\
&\xi_{1,0_4} = -1,62043139820, && \xi_{1,0_6} = -0,890328852651;\\
&\xi_{2,0_1} = 1,60281626452, && \xi_{2,0_2} = 0,408184155311, && \xi_{2,0_3} = 0,265945587541,\\
&\xi_{2,0_4} = 0,856971714096, && \xi_{2,0_6} = -1,00930692980;\\
&\xi_{3,0_1} = 0,958534529949, && \xi_{3,0_2} = 0,120316052745, && \xi_{3,0_3} = 1,02490924543,\\
&\xi_{3,0_4} = -0,370702322849, && \xi_{3,0_6} = 0,393900792945;\\
&\xi_{4,0_1} = -0,494744421722, && \xi_{4,0_2} = 0,00570200585236, && \xi_{4,0_3} = 1,02337140616,\\
&\xi_{4,0_4} = -0,190279173650, && \xi_{4,0_6} = -0,769749348792;\\
&\xi_{5,0_1} = 0,980319608250, && \xi_{5,0_2} = -1,689598303, && \xi_{5,0_3} = -0,967844689800,\\
&\xi_{5,0_4} = -1,91954474140, && \xi_{5,0_6} = -1,70725764039.
\end{aligned}$$

The external effect on each edge may be represented by decomposition

$$\begin{gathered}
NdW_j = A\sin(wt)\phi_1\varphi_1(x) + A\sin(wt)\phi_2\varphi_2(x) + A\sin(wt)\phi_3\varphi_3(x) + \\
-A\sin(wt)\phi_1\varphi_4(x) + A\sin(wt)\varphi_6(x)
\end{gathered}$$

with $A = 1.59459504258$, $w = 5$, $\phi_k = (\phi_{1_k}, \phi_{2_k}, \phi_{3_k}, \phi_{4_k}, \phi_{5_k})$ in accordance with the number of edges $j = 5$.

The generation of random variables included in the decomposition for random external action gave the following results:

$$\begin{aligned}
&\phi_{1_1} = 0,163047495059, && \phi_{1_2} = -1,63591146136, && \phi_{1_3} = -0,684145716945,\\
&\phi_{1_4} = 1,70539232724, && \phi_{1_5} = 1,62659212805;\\
&\phi_{2_1} = -1,64638499412, && \phi_{2_2} = -0,0658169964062, && \phi_{2_3} = 0,429340925424,\\
&\phi_{2_4} = -2,85293688880, && \phi_{2_5} = -0,390588206061;\\
&\phi_{3_1} = -1,39574691274, && \phi_{3_2} = -1,88806707585, && \phi_{3_3} = 0,954004456482,\\
&\phi_{3_4} = 0,899288864784, && \phi_{3_5} = -0,488478751168;\\
&\phi_{4_1} = 1,88421514070, && \phi_{4_2} = -0,243309185254, && \phi_{4_3} = -0,107823473285,\\
&\phi_{4_4} = 0,531389289132, && \phi_{4_5} = -2,06626981338;\\
&\phi_{5_1} = -0,735193065646, && \phi_{5_2} = -1,44835524134, && \phi_{5_3} = 0,247186393888,\\
&\phi_{5_4} = 3,64531828196, && \phi_{5_5} = -0,589661784796.
\end{aligned}$$

The system of differential equations contains five equations for $a_k(t)$. Obtained was one algebraic equation, the fifth one (since the fifth eigenvalue equals λ).

$$
\begin{aligned}
&50,0000000000\frac{da_1(t)}{dt}+\\
&+(6,94399999180\cos^4(t)-5,20799999385\cos^2(t)+0,433999999488)\sin(t)=0,\\
&26,0000000001\frac{da_2(t)}{dt}+0.399999999999a_2(t)+\\
&+(26,2748953373\cos^4(t)-19,7061715030\cos^2(t)+1,64218095857)\sin(t)=0,\\
&29,0000000001\frac{da_3(t)}{dt}+1,59999999999a_3(t)+\\
&+(-8,73495968431\cos^4(t)+6,55121976324\cos^2(t)-0,545934980270)\sin(t)=0,\\
&33,9999999999\frac{da_4(t)}{dt}+1,59999999999a_3(t)+\\
&+(-5,05641187458\cos^4(t)+3,79230890600\cos^2(t)-0,316025742165)\sin(t)=0,\\
&a_5(t)=0,\\
&41,0000000000\frac{da_6(t)}{dt}+6,40000000000a_6(t)+\\
&+(2,61619669064\cos^4(t)-1,96214751799\cos^2(t)+0,163512293166)\sin(t)=0.
\end{aligned}
$$

The following initial values are obtained for solving the Showalter – Sidorov problem

$$
\begin{aligned}
&a_1(0)=-0,392191341073,\\
&a_2(0)=0,116407043162,\\
&a_3(0)=0,374812563896,\\
&a_4(0)=-0,437183729135,\\
&a_5(0)=0,\\
&a_6(0)=-0,532669260320.
\end{aligned}
$$

Solutions are obtained

$$
\begin{aligned}
&a_1(t)=1,00000000000\cdot10^{-14}\cos(t)+0,001735999997\cos(5t)-0,393927341071,\\
&a_2(t)=-6,72917652624\cdot10^{-13}\cos(t)-5,63788659793\cdot10^{-12}\sin(t)-\\
&-8,01260979327\cdot10^{-14}\cos(3t)+4,10903066319\cdot10^{-16}\sin(3t)+\\
&+0,0126320416265\cos(5t)-0,388678203890\cdot10^{-4}\sin(5t)+\\
&+0,1103775001536e^{-0.0153846153845t},\\
&a_3(t)=4,29726397647\cdot10^{-14}\cos(t)-2,37090426285\cdot10^{-15}\sin(t)+\\
&-2,15444373418\cdot10^{-14}\cos(3t)-3,96219537316\cdot10^{-16}\sin(3t)-\\
&-0,376461045118\cdot10^{-2}\cos(5t)+0,415405291161\cdot10^{-4}\sin(5t)+\\
&+0,378577174347e^{-0,055172413782t},\\
&a_4(t)=3,63571037504\cdot10^{-13}\cos(t)-3,84957569124\cdot10^{-14}\sin(t)+\\
&+5,88813426710\cdot10^{-13}\cos(3t)-5,61584185793\cdot10^{-15}\sin(3t)-\\
&-0,185814168344\cdot10^{-2}\cos(5t)+0,393488827082cdot10^{-4}\sin(5t)+\\
&-0,435325587452e^{-0,105882352941t},\\
&a_5(t)=0,\\
&a_6(t)=-3,57151153337\cdot10^{-14}\cos(t)+5,575042393557\cdot10^{-15}\sin(t)-\\
&-2,02703237187\cdot10^{-14}\cos(3t)+1,05471603089\cdot10^{-15}\sin(3t)+\\
&+0,796844291651\cdot10^{-3}\cos(5t)-0,248770900808\cdot10^{-4}\sin(5t)+\\
&-0,533466104612e^{-0,156097560976t}.
\end{aligned}
$$

Next, the approximate solution to the Showalter – Sidorov problem is found on the graph in question. The analytical result is rather cumbersome; therefore, the results of the computational experiment are presented graphically (Figs. 2 – 3.) in the form of two-dimensional graphs at various points in time t^*. The abscissa axis reflects the values of the variable x and reflects the length of the edges of the graph. The ordinate axis reflects the values of the $\tilde{\zeta}_j(x, t^*)$ function: the dynamics of pressure and velocity of a viscoelastic incompressible fluid moving in the jth section of the pipeline with random external effect.

The colors show solutions on different edges of the graph: $\tilde{\zeta}_1(x, t^*)$ – blue, $\tilde{\zeta}_2(x, t^*)$ – green, $\tilde{\zeta}_3(x, t^*)$ – black, $\tilde{\zeta}_4(x, t^*)$ – red, $\tilde{\zeta}_5(x, t^*)$ – magenta.

The sequence of graphs reflects the development of the process over time, taking into account the structure of the graph.

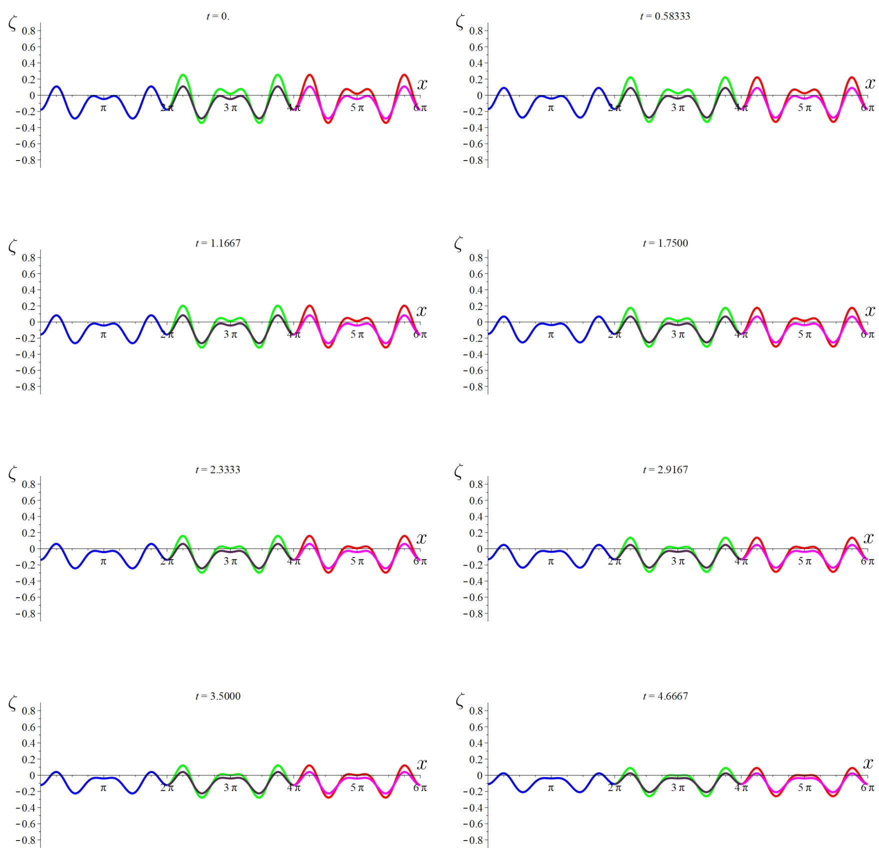

FIGURE 2. Results of computational experiment.

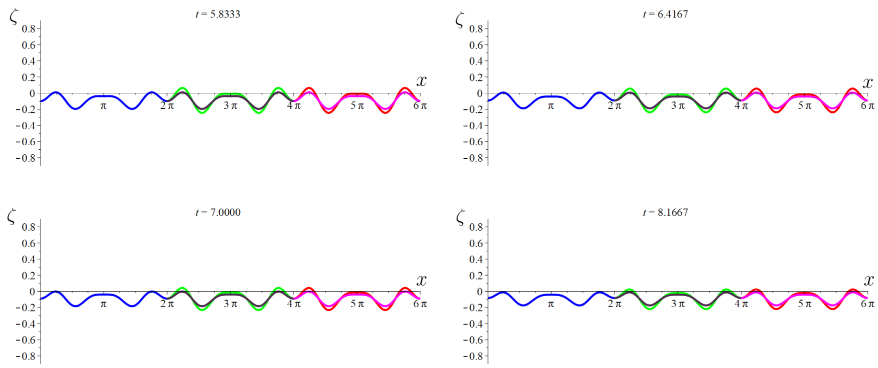

Figure 3. Results of computational experiment.

References

1. Brown, R.: A brief account of microscopical observations made in the months of June, July, and August, 1827, on the particles contained in the pollen of plants; and on the general existence of active molecules in organic and inorganic bodies, *Philosophical magazine* **4** (1828) 161–173.
2. Bachelier, L.: Theorie de la speculation, *Annales scientifiques de l'E.N.S.* **17** (1900) 21–86. (in French)
3. Einstein, A.: Uber die von der molekularkinetischen Theorie der Warme geforderte Bewegung von in ruhenden Flussigkeiten suspendierten Teilchen, *Annalen der Physik* **17** (1905) 549–560. (in German)
4. Favini, A., Sviridyuk, G.A. and Manakova, N.A.: Linear Sobolev Type Equations with Relatively p-Sectorial Operators in Space of "noises", *Abstract and Applied Analysis* **1** (2015) 1–8.
5. Favini, A., Sviridyuk, G.A. and Zamishlyaeva, A.A.: One class of Sobolev type equations of higher order with additive white noise", *Communications on Pure and Applied Analysis* **15**(1) (2016) 185–196.
6. Favini, A., Sviridyuk, G.A. and Sagadeeva, M.A.: Linear Sobolev Type equations with relatively p-Radial Operators in Space of "Noises", *Mediterranean Journal of Mathematics* **13**(6) (2016) 4607–4621.
7. Favini, A., Zagrebina, S.A. and Sviridyuk, G.A.: Multipoint initial-final value problems for dynamical Sobolev-type equations in the space of noises, *Electronic Journal of Differential Equations* **128** (2018) 1–10.
8. Matveeva, O.P. and Sukacheva, T.G.: *Mathematical models of viscoelastic incompressible liquids of nonzero order: monograph*, Chelyabinsk, Publishing Center of SUSU, 2014.
9. Oskolkov, A.P.: Nonlocal problems for one class of nonlinear operator equations that arise in the theory of Sobolev type equations, *Journal of Soviet Mathematics* **64**(1) (1993) 724–735.
10. Smoluchowski, M.V.: *Uber Unregelmassigkeitcn der Verteilung von Gasmolekulen und deren Einfluss auf Entropie und Zustandsgleichung*, W: "Boltzmann–Festschrift", Leipzig, 1904. (in German)
11. Soldatova, E.A. and Keller, A.V.: Numerical algorithm and computational experiments for one linear stochastic Hoff model, *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software (Bulletin SUSU MMCS)* **17**(2) (2024) 83–95.
12. Sviridyuk, G.A. and Manakova, N.A.: The Dynamical Models of Sobolev Type with Showolter–Sidorov Condition and Additive "Noise", *Bulletin of the South Ural State University. Series "Mathematical Modelling, Programming and Computer Software"* **7**(1) 2014 90–103. (in Russian)

13. Sviridyuk, G.A. and Shipilov, A.S.: Stability of solutions of Oskolkov linear equations on a geometric graph, *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki [J. Samara State Tech. Univ., Ser. Phys. Math. Sci.]* **2** (2009) 9–16. (in Russian)
14. Zagrebina, S.A. and Konkina, A.S.: Non–classical models of mathematical physics with a multipoint initial and final condition, *Bulletin of the South Ural State University. Ser. Mathematical Modelling, Programming and Computer Software* **15**(1) (2022) 60–83. (in Russian)
15. Zagrebina, S.A., Soldatova, E.A. and Sviridyuk, G.A.: The Stochastic Linear Oskolkov Model of the Oil Transportation by the Pipeline, *Springer Proceedings in Mathematics and Statistics* **113** (2015) 317–325.

Ekaterina Soldatova: Department of Mathematic and Computer Modeling, South Ural State University, Chelyabinsk, Chelyabinsk region 454008, Russia
Email address: soldatovaea@susu.ru

A.V. Keller: Department of Applied Mathematics and Mechanics, Voronezh State Technical University, Voronezh, Voronezh region 394006, Russia; Research Laboratory of the Non-classical Equations of Mathematical Physics, South Ural State University, Chelyabinsk, Chelyabinsk region 454008, Russia
Email address: alevtinak@inbox.ru