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OPTIMIZATION OF QUASI-LINEAR MODELS OF COMPLEX SYSTEMS WITH A FINITE NUMBER OF DETERMINISTIC PRIORITIES

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ABSTRACT. In this article, the authors present some results related to the search for an optimal solution to the optimization problem of managing an economic system with a finite number of interconnected institutions that are influenced by an external “regulator”. The regulator is interested in the fruitful interaction of all structures; it manages the system by setting appropriate priorities. The optimization problem comes down to finding the maximum point of the objective function of the regulator (optimizer). The paper presents an analytical solution to the problem and derives formulas for finding the maximum point of the objective function for the case of a finite number of deterministic priorities.

1. Introduction

This article is devoted to finding a solution to the optimization problem of managing an economic system with a finite number of interconnected institutions that are influenced by an external “optimizer.” An optimizer, interested in the successful functioning of all departments in total, manages the economic system under his control by prioritizing, making decisions based on expert assessments. In works [1]–[2], models with two and three priorities, giving a total of one, were proposed and studied. Further development of modeling of such systems is associated with the transition to a finite number of priorities and the transition to random priorities [3]–[7]. The question of the existence and uniqueness of a solution to an optimization problem in the case of a finite number of independent random priorities was solved in [3], but the final analytical result has not yet been obtained. This work is devoted to finding an optimal solution to the problem of managing an economic system with a finite number of deterministic priorities.

Let us introduce the basic notation and formulate the optimization problem. Let us denote multidirectional objective functions representing the requirements of participants in the economic system under study by $F_i(x_1, x_2, ...x_n), i = 1, 2, ..., m$. Let them $F_i(\overline{x}), \overline{x} \in R^n$ satisfy the following conditions: they are non-negative, non-zero, continuous, twice continuously differentiable on open sets $B_i = \{F_i > 0\}$

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and $\bigcap_{i=1}^{m} B_i \neq \emptyset$. According to the ideology proposed earlier in [2], we will represent them F_i as functions of a "quasi-linear" form:

$$F_i(x) = \left(\sum_{k=1}^{n} a_k^i x_k + b_i \right) \cdot I_{\left\{ \sum\limits_{k=1}^{n} a_k^i x_k + b_i > 0 \right\}}, i = 1, 2, ..., m. \tag{1.1}$$

Let us form the optimizer's objective function: $F = E(F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m})$, where $\alpha_i = \alpha_i(\omega)$ are independent random variables defined on the probability space $(\Omega, \mathcal{F}, P), P(0 \leq \alpha_i \leq 1) > 0$. The optimization task of the arbiter comes down to finding the maximum point of the function F. In [3], a criterion for the existence and uniqueness of the maximum point of the objective function was obtained, but the question of calculating the coordinates of this point remains open.

This article will consider solving an optimization problem in the case where the priorities are constant and satisfy the conditions: $\alpha_1 + \alpha_2 + .. + \alpha_m = 1$ and $\alpha_i > 0, i = 1, 2, ..., m$. Then the objective function of the arbiter will take the form:

$$F = F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m}. \tag{1.2}$$

Proposition 1.1. *Consider a system of vectors $\{\overline{a}^1, \overline{a}^2, ..., \overline{a}^m\}$, where $\overline{a}^i = (a_1^i, a_2^i, ..., a_n^i), i = 1, 2, ..., m$. Objective function F , which has the form 1.2, has a single stationary point in the region $x \in \bigcap_{i=1}^{m} B_i$, which is a global maximum point if the system of vector $\{\overline{a}^1, \overline{a}^2, ..., \overline{a}^m\}$ is linearly dependent and contains $m-1$ linearly independent vectors.*

The proof is trivial (follows from T.1[3]).

2. Main result

Let the function F of the form 1.2 satisfy the conditions of Proposition 1.1, then the optimization problem has a unique solution. Let's calculate the coordinates of the maximum point. Let's find the partial derivatives of the functions $F(x)$.

We have: $\frac{\partial F}{\partial x_k} = F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m} \left(\sum\limits_{i=1}^{m} \alpha_i a_k^i F_i^{-1} \right)$.

At a stationary point $\frac{\partial F}{\partial x_k} = 0, \forall k = 1, 2, ..., n$. Since $F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m} > 0$, then at a stationary point the equality must be satisfied:

$$\sum_{i=1}^{m} \alpha_i \overline{a}^i F_i^{-1} = 0. \tag{2.1}$$

Having transformed 2.1 we present the vector $\overline{a}^m$ in the following form:

$\overline{a}^m = - \sum\limits_{i=1}^{m-1} \frac{\alpha_i F_i^{-1}}{\alpha_m F_m^{-1}} \overline{a}^i = \sum\limits_{i=1}^{m-1} \frac{-\alpha_i c_i}{\alpha_m} \overline{a}^i$, where $c_i = \frac{F_m(x)}{F_i(x)} > 0$ (this designation was introduced in accordance with the ideology proposed in [2]). For brevity, we will use the notation $A_i = \frac{-\alpha_i c_i}{\alpha_m} < 0, i = 1, 2, ..., m-1$. Then

$\overline{a}^m = \sum\limits_{i=1}^{m-1} A_i \overline{a}^i$. At a stationary point A_i and c_i are constants.

Let us introduce the notation $s_i = \sum\limits_{k=1}^{m} a_k^i x_k$, then
$s_m = \sum\limits_{k=1}^{m} a_k^m x_k = \sum\limits_{k=1}^{m} \left(x_k \sum\limits_{i=1}^{m-1} A_i \overline{a}_i \right) = \sum\limits_{i=1}^{m-1} \left(A_i \sum\limits_{k=1}^{m} a_k^i x_k \right) = \sum\limits_{i=1}^{m-1} A_i s_i.$
Objective function 1.2 takes the form:

$$F(\overline{s}) = (s_1+b_1)^{\alpha_1}(s_2+b_2)^{\alpha_2}...(s_m+b_m)^{\alpha_m} = \prod_{j=1}^{m-1} (s_j+b_j)^{\alpha_j} \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{\alpha_m} . \quad (2.2)$$

Let's calculate the coordinates of the stationary point of the function . To do this, let's solve the system $\{ \frac{\partial F(\overline{s})}{\partial s_i} = 0, i = 1, 2, ..., m-1.$
We calculate

$$\frac{\partial F(\overline{s})}{\partial s_i} = \alpha_i (s_i + b_i)^{\alpha_i - 1} \prod_{j=1, j \neq i}^{m-1} (s_j + b_j)^{\alpha_j} \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{\alpha_m} +$$
$$+ \prod_{j=1}^{m-1} (s_j + b_j)^{\alpha_j} \alpha_m A_i \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{\alpha_m - 1} =$$
$$= \prod_{j=1}^{m-1} (s_j + b_j)^{\alpha_j} \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{\alpha_m} \left(\alpha_i (s_i + b_i)^{-1} + \alpha_m A_i \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{-1} \right).$$

Considering that $\alpha_m A_i = -\alpha_i c_i$, we get $\forall i = 1, 2, ..., m-1$:

$$\frac{\partial F(\overline{s})}{\partial s_i} = \prod_{j=1}^{m-1} (s_j + b_j)^{\alpha_j} \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{\alpha_m} \left(\alpha_i (s_i + b_i)^{-1} - \alpha_i c_i \left(\sum_{i=1}^{m-1} A_i s_i + b_m \right)^{-1} \right).$$

Because $F(\overline{s}) > 0$, then $\{ \frac{\partial F(\overline{s})}{\partial s_i} = 0 \Leftrightarrow$
$\left\{ \alpha_i (s_i + b_i)^{-1} - \alpha_i c_i \left(\sum\limits_{i=1}^{m-1} A_i s_i + b_m \right)^{-1} = 0 \right., \forall i = 1, 2, ..., m-1.$
Considering that $\alpha_i > 0$, we have $\left\{ \sum\limits_{i=1}^{m-1} A_i s_i + b_m = c_i (s_i + b_i) \right., \forall i = 1, 2, ..., m-1.$
Let's transform the system:

$$\begin{cases} (A_1 - c_1) s_1 + A_2 s_2 + ... + A_{m-1} s_{m-1} = c_1 b_1 - b_m, \\ A_1 s_1 + (A_2 - c_2) s_2 + ... + A_{m-1} s_{m-1} = c_2 b_2 - b_m, \\ ... \\ A_1 s_1 + A_2 s_2 + ... + (A_{m-1} - c_{m-1}) s_{m-1} = c_{m-1} b_{m-1} - b_m. \end{cases}$$

Let us calculate the main determinant of the system:

$$\Delta=(A_1-c_1)(-1)^m c_2\ldots c_{m-1}+c_1A_2(-1)^m c_3\ldots c_{m-1}+\ldots+c_1A_{m-1}(-1)^m c_2\ldots c_{m-2}=$$

$$=(-1)^m\prod_{j=1}^{m-1}c_j\left(\frac{A_1-c_1}{c_1}+\frac{A_2}{c_2}+...+\frac{A_{m-1}}{c_{m-1}}\right)=$$

$$=(-1)^m\prod_{j=1}^{m-1}c_j\left(\frac{A_1}{c_1}+\frac{A_2}{c_2}+...+\frac{A_{m-1}}{c_{m-1}}-1\right).$$

Considering that $\frac{A_i}{c_i}=\frac{-\alpha_i}{\alpha_m}$, $i=1,2,...,m-1$, we get:

$$\Delta=(-1)^m\prod_{j=1}^{m-1}c_j\left(-\frac{\alpha_1}{\alpha_m}-\frac{\alpha_2}{\alpha_m}-...-\frac{\alpha_{m-1}}{\alpha_m}-1\right)=$$

$$=\frac{(-1)^{m-1}}{\alpha_m}\prod_{j=1}^{m-1}c_j(\alpha_1+\alpha_2...+\alpha_m)=\frac{(-1)^{m-1}}{\alpha_m}\prod_{j=1}^{m-1}c_j.$$

Since $\alpha_m>0$ both are constants $c_i>0$, then $\Delta\neq 0$.

Let us calculate the determinants $\Delta_i, i=1,2,...m-1$.

$$\Delta_i=(c_ib_i-b_m)(-1)^m\prod_{j=1,j\neq i}^{m-1}c_j+(c_1b_1-c_ib_i)A_1(-1)^m\prod_{j=2,j\neq i}^{m-1}c_j+...$$

$$+(c_{i-1}b_{i-1}-c_ib_i)A_{i-1}(-1)^m\prod_{\substack{j=1\\ j\neq i-1,i}}^{m-1}c_j+(c_{i+1}b_{i+1}-c_ib_i)A_{i+1}(-1)^m\prod_{\substack{j=1\\ j\neq i,i+1}}^{m-1}c_j+...$$

$$+(c_{m-1}b_{m-1}-c_ib_i)A_{m-1}(-1)^m\prod_{j=1,j\neq i}^{m-2}c_j=$$

$$=(-1)^m\prod_{j=1}^{m-1}c_j\left(\frac{c_ib_i-b_m}{c_i}+\sum_{k=1,k\neq i}^{m-1}(c_kb_k-c_ib_i)\frac{A_k}{c_ic_k}\right).$$

Since $\frac{A_i}{c_i}=\frac{-\alpha_i}{\alpha_m}$, $i=1,2,...,m-1$, we get:

$$\Delta_i=\frac{(-1)^{m-1}}{\alpha_mc_i}\prod_{j=1}^{m-1}c_j\left((b_m-c_ib_i)\alpha_m+\sum_{k=1}^{m-1}(c_kb_k-c_ib_i)\alpha_k\right).$$

Then the solution of the system has the form:

$$s_i=\frac{1}{c_i}\left((b_m-c_ib_i)\alpha_m+\sum_{k=1}^{m-1}(c_kb_k-c_ib_i)\alpha_k\right),\forall i=1,2,...,m-1. \qquad (2.3)$$

Theorem 2.1. *Let the function $F(\overline{s})$, which has the form 2.2, satisfy the conditions of 1.1. Then the coordinates of the global maximum point $M(s_1,s_2,...,s_{m-1})$ of the objective function $F(\overline{s})$ are calculated using the formulas 2.3.*

3. Example

Let us consider the following case $m = 3$. Considering that $\alpha_3 = 1 - (\alpha_1 + \alpha_2)$, we get:

$$s_1 = \frac{1}{c_1}\left((b_3 - c_1b_1)(1 - (\alpha_1 + \alpha_2)) + (c_2b_2 - c_1b_1)\alpha_2\right) =$$

$$= \frac{1}{c_1}\left((b_3 - c_1b_1)(1 - \alpha_1) - b_3\alpha_2 + c_1b_1\alpha_2 + c_2b_2\alpha_2 - c_1b_1\alpha_2\right) =$$

$$= \frac{1}{c_1}\left((\alpha_1 - 1)(c_1b_1 - b_3) + \alpha_2(c_2b_2 - b_3)\right);$$

$$s_2 = \frac{1}{c_2}\left((b_3 - c_2b_2)(1 - (\alpha_1 + \alpha_2)) + (c_1b_1 - c_2b_2)\alpha_1\right) =$$

$$= \frac{1}{c_2}\left(\alpha_1(c_1b_1 - b_3) + (\alpha_2 - 1)(c_2b_2 - b_3)\right);$$

This result completely coincides with the result obtained in [2].

References

1. Vagin V.S., Pavlov I.V.: Modeling and optimization of quasi-linear complex systems taking into account the probabilistic nature of priorities, *Scientific and technical journal "Vestnik RGUPS" – Rostov-on-Don* **1** (61) (2016) 135–139.(in Russian)
2. Volosatova T.A., Danekyants A.G.: Optimization of quasilinear complex systems: the case of three deterministic priorities, *International scientific research journal* **10-2**(52) (2016) 127–132.(in Russian)
3. Pavlov I.V., Uglich S.I. : Optimization of complex systems of quasi-linear type with several independent priorities, *Bulletin of the Rostov State Transport University* **67**(3) (2019) 140–145.(in Russian)
4. Volosatova T.A., Danekyants A.G. : Modeling of quasilinear complex systems: the case of three probabilistic priorities with unit sum, *Theory of Probability and its Applications* **62**(4) (2018) 647–648.
5. Volosatova T.A., Danekyants A.G. : Optimization of quasilinear complex systems in the case of special objective functions with dependent priorities , *Bulletin of the Rostov State Transport University* **73**(1) (2019) 135–142.(in Russian)
6. Volosatova T.A., Neumerzhitskaya N.V., Uglich S.I.,: Strict convexity of the objective function and uniqueness of the maximum point in a model with three arbitrary random priorities, *Journal of Physics: Conference Series* **224** (2020) 01014.
7. Volosatova T.A., Neumerzhitskaya N.V., Uglich S.I., Pavlov I.V.: Strict convexity of the objective function and uniqueness of the maximum point in a model with three arbitrary random priorities, *Journal of Physics: Conference Series* **2131**(3) (2021) 032001.

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