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PRIMENESS IN SEMINEARRINGS AND S-SEMIGROUPS

PRAKASH PADOOR, BABUSHRI SRINIVAS KEDUKODI, SYAM PRASAD KUNCHAM AND KAVITHA KOPPULA*

Abstract. In this paper, we prove results on various prime ideals of seminearring, which is a generalization of nearring. We obtain the relationship among various prime ideals of S-semigroup, which is a module over seminearring. In addition, we prove one-one correspondence between the strong ideals of S-semigroup R and the strong ideals of homomorphic image of R.

1. Introduction

The notion of primeness plays an important role in studying algebraic structures. Holcombe [5] extended the definition of prime rings to nearrings and characterized 0-prime, 1-prime and 2-prime nearrings as three different types of prime nearrings within the class of all nearrings. Subsequently, Groenewald's [4] introduction of the 3-prime nearring further broadened the scope of the study. Notably, these findings have substantial implications for studying nearrings and their properties. Furthermore, Booth et al. [2] presented the idea of equiprime, a generalization of prime rings to nearrings. Later on, Veldsman [13] presented related results.

The idea of $\nu-$prime $(\nu = 3, c, e)$ ideals was extended to N-groups and related results were provided by several authors, including Booth et al. [3], Juglal et al. [6] and Taşdemir et al. [12]. We refer to Pilz [9], Bhavanari, and Kuncham [1] for the results and isomorphism theorems on nearrings and N-groups.

Koppula et al. [7] defined and proved results on the ideal of seminearring and discussed fundamental properties. Different prime strong ideals of seminearring and their corresponding prime radicals were described by Koppula et al. [8]. Prakash et al. [10] proved classical isomorphism theorems in S-semigroups. Further, different prime strong ideals of S-semigroup and the relationship between different prime strong ideals of S-semigroup along with suitable examples were provided by Prakash et al.[11].

This paper discusses some properties of different prime strong ideals of seminearring and their interrelation. Additionally, we obtain results on these strong ideals by providing the appropriate and adequate conditions. Further, we prove the one-to-one correspondence between the strong ideal of S-semigroup, a module over a seminearring, and the strong ideals of it's homomorphic image.

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* Corresponding Author.

2. Preliminaries

This section discusses some fundamental definitions and findings that help in obtaining the results of the present paper.

Definition 2.1. [14] A non-empty set S with respect to the binary operations $+$ and $\cdot$ is said to be a right seminearring if

1. $(S,+)$ is a semigroup.
2. There exists $0 \in S$ such that $0+s=s+0=s,\ \forall\ s \in S$.
3. $(S,\cdot)$ is a semigroup.
4. For all $u,y,e \in S,\ (u+y)e = ue+ye$.
5. For all $a \in S, 0a = 0$.

Definition 2.2. Let T be a nonempty subset of S. For $u,v \in S,\ \ u \equiv_T v$ if and only if there exist $l_1, l_2 \in T$ such that $l_1 + u = l_2 + v$.

In the following, we consider S as a right seminearring.

Definition 2.3. [7] A nonempty subset X of S is said to be a strong ideal of S if the below mentioned conditions are satisfied :

1. For $e,f \in X, e+f \in X$
2. For $s \in S, s+X \subseteq X+s$
3. If $e \equiv_X f$, then $e \in X+f,\ e,f \in S$
4. $e(X+f) \subseteq X+ef$ for all $e,f \in S$
5. $Xe \subseteq X$ for all $e \in S$

Definition 2.4. [10] Let $(S,+,\cdot)$ be a seminearring and $(R,+)$ be a semigroup. Then R is said to be a S-semigroup, if there exists a mapping $*: S \times R \to R$ defined as $*(w,\kappa) \to w*\kappa$ satisfies the below conditions

1. $(q+v)*\kappa = q*\kappa + v*\kappa$
2. $(q \cdot v)*\kappa = q*(v*\kappa)$
3. $0*\kappa = 0$,

for all $\kappa \in R$ and $q,v \in S$.

In the following, we denote $w*\kappa$ by $w\kappa$.

Definition 2.5. [10] A non-empty subset T of a S-semigroup R is said to be a strong ideal of R if the below mentioned conditions are satisfied.

1. If $e,f \in T$ then $e+f \in T$.
2. $e+T \subseteq T+e,\ \forall\ e \in R$.
3. For $e,f \in R$, if $e \equiv_T f$ then $e \in T+f$.
4. $s(T+e) \subseteq T+se\ \ \forall\ s \in S,\ e \in R$.

Definition 2.6. A subset $\phi \neq M$ of a semigroup $(S,+)$ is said to be a subsemigroup, if $q,v \in M$, then $q+v \in M$.

Definition 2.7. A subsemigroup Δ of a S-semigroup R is a S-subsemigroup if $S\Delta \subseteq \Delta$ and $0 \in \Delta$.

Definition 2.8. [10] Let R and R' be S-semigroups. Then a mapping $\varrho : R \longrightarrow R'$ is said be a S-homomorphism (or S-semigroup homomorphism) if

(1) $\varrho(r_1 + r_2) = \varrho(r_1) + \varrho(r_2)$
(2) $\varrho(sr_1) = s\varrho(r_1)$, for all $s \in S$, $r_1, r_2 \in R$.

Definition 2.9. [10] A S-homomorphism $\varrho : R \longrightarrow R'$ is said to be a strong S-homomorphism if $\varrho(x) = \varrho(y)$, then $x \in ker\varrho + y$.

Definition 2.10. [11] Let S be a seminearring, R is any S-semigroup and K is a strong ideal of R such that $SR \nsubseteq K$. Then K is called

(1) **3-prime** strong: for $a \in S, r \in R$, if $aSr \subseteq K$, then $aR \subseteq K$ or $r \in K$.
(2) Completely prime (**c-prime**) strong: for $a \in S, r \in R$, if $ar \in K$, then $aR \subseteq K$ or $r \in K$.
(3) Equiprime (**e-prime**) strong: for $a \in S, r_1, r_2 \in R$, if $asr_1 \equiv_K asr_2$ for all $s \in S$, then $aR \subseteq K$ or $r_1 \equiv_K r_2$.
(4) Completely equiprime (**c-e-prime**) strong: for $a \in S$, $r_1, r_2 \in R$, if $ar_1 \equiv_K ar_2$, then $aR \subseteq K$ or $r_1 \equiv_K r_2$.

3. Primeness in Seminearrings and S-semigroups

In this section, we prove 1-1 correspondence between the strong ideals of S-semigroup R and the strong ideals of S-semigroup $R^{'}$. In addition, we obtain the results on various prime ideals of S-semigroups.

Theorem 3.1. If $\pi : R \longrightarrow R^{'}$ is an onto strong S-homomorphism, then π induces 1-1 correspondence between the strong ideals of R containing $Ker\pi$ and the strong ideals of $R^{'}$.

Proof. Let Q be a strong ideal of R containing $Ker\pi$.
Then we have to show that $\pi(Q) = \{\pi(q) \mid q \in Q\}$ is a strong ideal of $R^{'}$.

Let $e, f \in \pi(Q)$. Then $\pi(e') = e,\ \pi(f') = f$ for some $e', f' \in Q$.
Now, $e + f = \pi(e') + \pi(f') = \pi(e' + f')$. As $e', f' \in Q$, we have $e' + f' \in Q$.
This implies $\pi(e' + f') \in \pi(Q) \implies e + f \in \pi(Q)$.

Let $s' \in R^{'}$. Now, take $x \in s' + \pi(Q)$. Then $x = s' + \pi(q)$ for some $\pi(q) \in \pi(Q)$.
As $s' \in R^{'}$ and π is a S-homomorphism, $\pi(s) = s'$ for some $s \in R$. Then,

$$\begin{aligned} x = s' + \pi(q) = \pi(s) + \pi(q) = \pi(s + q) &= \pi(q'' + s), \text{ for some } q'' \in Q. \\ &= \pi(q'') + \pi(s) \\ &\in \pi(Q) + s'. \end{aligned}$$

Let $u', v' \in R^{'}$ be such that $u' \equiv_{\pi(Q)} v'$.
Then $q' + u' = q'' + v'$ for some $q', q'' \in \pi(Q)$.
As $q', q'' \in \pi(Q)$, we have $q' = \pi(q_1)$ and $q'' = \pi(q_2)$ for some $q_1, q_2 \in Q$.
As $u', v' \in R^{'}$, $\pi(u) = u'$ and $\pi(v) = v'$ for some $u, v \in R$.
Now, $q' + u' = q'' + v'$ implies $\pi(q_1) + \pi(u) = \pi(q_2) + \pi(v)$.
This implies $\pi(q_1 + u) = \pi(q_2 + v)$.
As π is strong, we get $(q_1 + u) \in Ker\pi + (q_2 + v)$.
Then there exists $p_1 \in Ker\pi$ such that $q_1 + u = p_1 + (q_2 + v)$.
As $Ker\pi \subseteq Q$, we get $q_1 + u = q_3 + v$ $(p_1 + q_2 = q_3 \in Q)$.

This gives $u \equiv_Q v$. This implies $u \in Q + v$.
Then $u = q_4 + v$ for some $q_4 \in Q$.
$\implies \pi(u) = \pi(q_4 + v) = \pi(q_4) + \pi(v)$
That is, $u' = \pi(q_4) + v' \in \pi(Q) + v'$.

Let $s \in S$ and $r' \in R^{'}$. Now, take $z \in s(\pi(Q) + r')$.
Then $z = s(\pi(q_1) + r^{'})$ for some $\pi(q_1) \in \pi(Q)$ and $q_1 \in Q$.
As $r' \in R^{'}$, $r' = \pi(r)$ for some $r \in R$. Now,

$$\begin{aligned} z = s(\pi(q_1) + \pi(r)) = s(\pi(q_1 + r)) &= \pi[s(q_1 + r)] \\ &= \pi(q_2 + sr), \text{for some } q_2 \in Q. \\ &= \pi(q_2) + \pi(sr) \\ &= \pi(q_2) + s\pi(r) \\ &= \pi(q_2) + sr' \in \pi(Q) + sr'. \end{aligned}$$

Hence $s(\pi(Q) + r') \subseteq \pi(Q) + sr'$.
Therefore $\pi(Q)$ is a strong ideal of $R^{'}$.
Conversely, let J be a strong ideal of $R^{'}$.
Then we show that $\pi^{-1}(J) = \{x \in R \mid \pi(x) \in J\}$ is a strong ideal of R.
Let $e, f \in \pi^{-1}(J)$. Then $e, f \in R$ and $\pi(e), \pi(f) \in J$.
$\implies \pi(e) + \pi(f) \in J$, because J is a strong ideal of $R^{'}$.
$\implies \pi(e + f) \in J$
$\implies e + f \in \pi^{-1}(J)$.

Let $s \in R$. Now, take $a \in s + \pi^{-1}(J)$. Then $a = s + e$ for some $e \in \pi^{-1}(J)$.
Now, $\pi(a) = \pi(s + e) = \pi(s) + \pi(e)$.
As J is a strong ideal of $R^{'}$, $\pi(a) = \pi(f) + \pi(s)$ for some $\pi(f) \in J$.
$\implies \pi(a) = \pi(f+s) \implies a \in Ker\pi + (f+s)$ (since π is strong homomorphism).
$\implies a = p_1 + f + s$ for some $p_1 \in Ker\pi$.
Now, $\pi(p_1 + f) = \pi(p_1) + \pi(f) = \pi(0) + \pi(f) = \pi(0 + f) = \pi(f) \in J$
That is, $\pi(p_1 + f) \in J$ and we have $p_1 + f \in R \implies p_1 + f \in \pi^{-1}(J)$.
Then $a = (p_1 + f) + s \in \pi^{-1}(J) + s$.
Hence $s + \pi^{-1}(J) \subseteq \pi^{-1}(J) + s$.
Let $e, f \in R$ be such that $e \equiv_{\pi^{-1}(J)} f$.
Then $a + e = b + f$ for some $a, b \in \pi^{-1}(J)$.
$a, b \in \pi^{-1}(J) \implies \pi(a), \pi(b) \in J$ and $a, b \in R$.
Then $\pi(a + e) = \pi(b + f) \implies \pi(a) + \pi(e) = \pi(b) + \pi(f)$.
$\implies \pi(e) \equiv_J \pi(f) \implies \pi(e) \in J + \pi(f)$
$\implies \pi(e) = \pi(j_1) + \pi(f)$, for some $\pi(j_1) \in J$ and $j_1 \in R$
$\quad = \pi(j_1 + f)$
As π is a strong S-homomorphism, we get $e \in Ker\pi + (j_1 + f)$.
$\implies e = (p_1 + j_1) + f$ for some $p_1 \in Ker\pi$.
Now, $\pi(p_1 + j_1) = \pi(p_1) + \pi(j_1) = \pi(0) + \pi(j_1) = \pi(0 + j_1) = \pi(j_1) \in J$.
That is, $\pi(p_1 + j_1) \in J$ and we have $p_1 + j_1 \in R \implies p_1 + j_1 \in \pi^{-1}(J)$.
Then $e = (p_1 + j_1) + f \in \pi^{-1}(J) + f$.
Let $s \in S$ and $r \in R$. Now, take $z \in s(\pi^{-1}(J) + r)$.

Then there exists $a \in \pi^{-1}(J)$ such that $z = s(a+r)$. Now,

$$\pi(z) = \pi(s(a+r)) = s\pi(a+r) = s(\pi(a)+\pi(r)) \in J + s\pi(r).$$

This implies $\pi(z) = \pi(j_1) + s\pi(r)$, for some $\pi(j_1) \in J$ and $j_1 \in R$
$= \pi(j_1) + \pi(sr) = \pi(j_1 + sr)$.
As π is a strong S-homomorphism, we get $z \in Ker\pi + (j_1 + sr)$.
$\implies z = p_1 + j_1 + sr$, for some $p_1 \in Ker\pi$.
Now, $\pi(p_1 + j_1) = \pi(p_1) + \pi(j_1) = \pi(0) + \pi(j_1) = \pi(0 + j_1) = \pi(j_1) \in J$
That is, $\pi(p_1 + j_1) \in J$ and we have $p_1 + j_1 \in R \implies p_1 + j_1 \in \pi^{-1}(J)$.
Then $z = (p_1 + j_1) + sr \in \pi^{-1}(J) + sr$.
Hence $s(\pi^{-1}(J) + r) \subseteq \pi^{-1}(J) + sr$ for all $s \in S, r \in R$.
Therefore $\pi^{-1}(J)$ is a strong ideal of R.
Let $c \in Ker\pi$. Then $\pi(c) = \pi(0) \in J$.
$\implies c \in \pi^{-1}(J) \implies Ker\pi \subseteq \pi^{-1}(J)$.
Hence $\pi^{-1}(J)$ is a strong ideal of R containing $Ker\pi$.

Let $\wp$ be the set of all strong ideals of R containing $Ker\pi$ and $\wp^{'}$ be the set of all strong ideals of $R^{'}$. Now, $h : \wp \longrightarrow \wp^{'}$ is defined as $h(P) = \{\pi(x) \mid x \in P\} = \pi(P)$, for all $P \in \wp$ and $\sigma : \wp^{'} \longrightarrow \wp$ is defined as $\sigma(J) = \{p \in R \mid \pi(p) \in J\} = \pi^{-1}(J)$, for all $J \in \wp^{'}$.
Now, $(\sigma \circ h)(P) = \sigma(h(P)) = \sigma(\pi(P)) = \pi^{-1}(\pi(P)) = (\pi^{-1} \circ \pi)(P) = P$
Similarly, $(h \circ \sigma)(P) = h(\sigma(P)) = h(\pi^{-1}(P)) = \pi(\pi^{-1}(P)) = (\pi \circ \pi^{-1})(P) = P$
Therefore $(h \circ \sigma)(P) = (\sigma \circ h)(P) = P$

This implies $h^{-1} = \sigma$ and h, σ are bijective functions.
Thus the one-one correspondence is established.

Definition 3.1. Let S be a seminearring and K be a strong ideal of S. Then K is called:

(1) **3-prime** strong [8]: for $q, b \in S$, if $qSb \subseteq K$, then $q \in K$ or $b \in K$.
(2) **c-prime** strong [8]: for $q, b \in S$, if $qb \in K$, then $q \in K$ or $b \in K$.
(3) **e-prime** strong [8]: for $q, x, y \in S$, if $qsx \equiv_K qsy$ for all $s \in S$, then $q \in K$ or $x \equiv_K y$.
(4) **c-e-prime** strong [11]: for $q, x, y \in S$, if $q \notin K$ and $qx \equiv_K qy$, then $x \equiv_K y$.

Remark 1. If K is a c-prime strong ideal of seminearring S and if $xa \equiv_K ya$ for $x, y, a \in S$, then we assume $x \equiv_K y$ or $a \in K$.
The above condition holds in case of nearrings obviously as follows.
If $xa \equiv_K ya$, then $xa - ya \in K \implies (x-y)a \in K$
$\implies x - y \in K$ or $a \in K$, since K is c-prime.

Definition 3.2. A seminearring S is said to be right (respectively, left) permutable seminearring if $qwe = qew$ (respectively, $qwe = wqe$) $\forall\, q, w, e \in S$.
A seminearring S is said to be permutable seminearring if it is both left and right permutable.

Definition 3.3. $S_d = \{d \in S \mid d(a+b) = da + db,\ \forall\, a, b \in S\}$.

Proposition 3.1. If K is a c-e-prime strong ideal of a zero-symmetric seminearring S, then K is a c-prime strong ideal of S.

Proof. Suppose that K is c-e-prime strong and $q, e \in S$ be such that $qe \in K$.
Assume that $q \notin K$. Because $qe \in K$, we get $qe \equiv_K 0 \implies qe \equiv_K q0$ ($q0 = 0$ in a zero-symmetric seminearring).
As K is c-e-prime and $q, e \in S$, we get $e \equiv_K 0 \implies e \in K$.
Thus K is a c-prime strong ideal of S.

Definition 3.4. A S-semigroup Γ is said to be monogenic if there exists $\gamma \in \Gamma$ which satisfies $S\gamma = \Gamma$, where $S\gamma = \{x\gamma : x \in S\}$.

Remark 2. [11] Let S be a seminearring and Γ be a S-semigroup and P be a strong ideal of Γ.
For $s, z \in S$, if $z\gamma \equiv_P s\gamma \ \forall \ \gamma \in \Gamma$, then we assume $z \equiv_{(P:\Gamma)} s$.

If P is a strong ideal of Γ, we denote it as $P \lhd \Gamma$.

Proposition 3.2. If S is a zero symmetric seminearring, Γ is a S-semigroup and $P \lhd \Gamma$, then $SP \subseteq P$.

Proof. As S is zero symmetric, we have $s0_\Gamma = 0_\Gamma \ \forall \ s \in S$.
Because $P \lhd \Gamma$, we have $sp = s(p + 0_\Gamma) \in P + s0_\Gamma$, $\forall \ s \in S, p \in P$. Hence $SP \subseteq P$.

Definition 3.5. Let Q and B be any two subsets of S-semigroup R. Then $(Q : B) = \{s \in S \mid sB \subseteq Q\}$.

Proposition 3.3. If K is a c-prime ideal of S-semigroup R, then $(K : R)$ is a c-prime ideal of seminearring S.

Proof. Let $x, y \in S$ be such that $xy \in (K : R)$ and assume that $y \notin (K : R)$.
Then $yR \nsubseteq K$, which implies $y\gamma_0 \notin K$ for some $\gamma_0 \in R$.
We have $xy \in (K : R)$, which implies $(xy)R \subseteq K \implies xy\gamma_0 \in K \ \forall \ \gamma_0 \in R$.
$\implies x(y\gamma_0) \in K$.
Since K is c-prime and $y\gamma_0 \notin K$, we get $xR \subseteq K$, which implies $x \in (K : R)$.
Thus $(K : R)$ is a c-prime ideal of S.

Proposition 3.4. If S is a permutable zero symmetric seminearring, then K is a c-prime strong ideal of monogenic S-semigroup R if and only if K is 3-prime strong.

Proof. Assume that S is permutable and K is 3-prime strong. Let $s \in S, \gamma \in R$ be such that $s\gamma \in K$.
Then $Ss\gamma \subseteq SK \subseteq K$.
Since R is monogenic, $S\gamma_0 = R$, for some $\gamma_0 \in R$.
Hence $x\gamma_0 = \gamma$, for some $x \in S, \ \gamma \in R$.
Since S is left permutable, $Ss\gamma = Ssx\gamma_0 = sSx\gamma_0 = sS\gamma \subseteq K$.
Since K is 3-prime strong, either $sR \subseteq K$ or $\gamma \in K$.
Thus K is c-prime strong.
Converse follows trivially. i.e; K is c-prime strong $\implies$ K is 3-prime strong.

Remark 3. Let K be a c-prime strong ideal of S-semigroup R and $x_1, x_2 \in S, \gamma_0 \in R$ be such that $x_1\gamma_0 \equiv_K x_2\gamma_0$. Then we assume $x_1\gamma \equiv_K x_2\gamma$ for all $\gamma \in R$

or $\gamma_0 \in R$.
The above condition holds good in nearrings obviously.
Let $x_1\gamma_0 \equiv_K x_2\gamma_0 \implies x_1\gamma_0 - x_2\gamma_0 \in K \implies (x_1 - x_2)\gamma_0 \in K$.
If K is c-prime, then $(x_1 - x_2)R \subseteq K$ or $\gamma_0 \in K$.

Proposition 3.5. If S is a permutable seminearring and K is a strong ideal of monogenic S-semigroup R, then K is an e-prime strong ideal of R if and only if K is a c-prime strong ideal of R.

Proof. First, we assume that K is c-prime. Let $a \in S,\ \gamma_1, \gamma_2 \in R$ be such that $as\gamma_1 \equiv_K as\gamma_2\ \forall s \in S$. We need to show that either $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
Since R is monogenic, there exists $\gamma_0 \in R$ which satisfies $S\gamma_0 = R$.
Then $\gamma_1 = x\gamma_0,\ \gamma_2 = y\gamma_0$ for some $x, y \in S$. Let $s \in S$ be arbitrarily fixed.
Then, $as\gamma_1 \equiv_K as\gamma_2 \implies asx\gamma_0 \equiv_K asy\gamma_0 \implies sax\gamma_0 \equiv_K say\gamma_0$ (since S is left permutable).
Since K is c-prime, $sax\gamma \equiv_K say\gamma\ \forall\ \gamma \in R$ or $\gamma_0 \in K$(by Remark 3).
$\implies sax \equiv_{(K:R)} say$, by Remark 2.
For $q \in S \setminus (K : R)$, we get $saxq \equiv_{(K:R)} sayq$
$\implies sxaq \equiv_{(K:R)} syaq$, since right permutable.
$\implies sxa \equiv_{(K:R)} sya$ or $q \in (K : R)$, since $(K : R)$ is c-prime.
$sxa \equiv_{(K:R)} sya \implies sx \equiv_{(K:R)} sy$ or $a \in (K : R)$, since $(K : R)$ is c-prime.
If $a \in (K : R)$, then $aR \subseteq K$ and thus K is equiprime strong ideal of S-semigroup R.
Suppose otherwise, $sx \equiv_{(K:R)} sy$, then $sxq \equiv_{(K:R)} syq$, since $(K : R)$ is an ideal.
$sxq \equiv_{(K:R)} syq \implies xsq \equiv_{(K:R)} ysq$, since left permutable.
Since $q \notin (K : R)$ and $(K : R)$ is c-prime, we have $xs \equiv_{(K:R)} ys$.
Again since $(K : R)$ is c-prime, $x \equiv_{(K:R)} y$ or $s \in (K : R)$.
If let $s \in (K : R)$, then $sR \subseteq K$, a contradiction to $sR \not\subseteq K$.
$x \equiv_{(K:R)} y$ implies $i_1 + x = i_2 + y$, for some $i_1, i_2 \in (K : R)$.
$(i_1 + x)\gamma_0 = (i_2 + y)\gamma_0 \implies i_1\gamma_0 + x\gamma_0 = i_2\gamma_0 + y\gamma_0$, by S-semigroup definition.
Since $i_1 \in (K : R)$, then $i_1\gamma \in K, \forall\ \gamma \in R \implies i_1\gamma_0 \in K$. Similarly $i_2\gamma_0 \in K$.
$x\gamma_0 \equiv_K y\gamma_0 \implies \gamma_1 \equiv_K \gamma_2$.
Hence K is an equiprime strong ideal of S-semigroup R.
On the other hand, if K is equiprime, then K is 3-prime.
If S is permutable seminearring, then K is c-prime if and only if K is 3-prime, by Proposition 3.4.
This implies K is equiprime. Then K is c-prime ideal of S-semigroup.

Proposition 3.6. If K is an equiprime strong ideal of S-semigroup R, then $(K : R)$ is an equiprime strong ideal of S.

Proof. Let K be an e-prime strong ideal of R and let $a, x, y \in S$ be such that $asx \equiv_{(K:R)} asy\ \forall\ s \in S$. Now, fix $s \in S$ arbitrarily. Then there exist $i_1, i_2 \in (K : R)$ such that $i_1 + asx = i_2 + asy$.
Let $\gamma \in R$. Then $(i_1 + asx)\gamma = (i_2 + asy)\gamma$. This implies $i_1\gamma + asx\gamma = i_2\gamma + asy\gamma$.
Then $asx\gamma \equiv_K asy\gamma$.
Let $\gamma \in R$ be fixed. Then $x\gamma = \gamma_1, y\gamma = \gamma_2$
This gives $as\gamma_1 \equiv_K as\gamma_2$

As $s \in S$ is arbitrary, we get $as\gamma_1 \equiv_K as\gamma_2 \ \forall \ s \in S$
Since K is e-prime strong, either $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
If $aR \subseteq K$, then $a \in (K : R)$ and thus $(K : R)$ is e-prime strong.
If $\gamma_1 \equiv_K \gamma_2$, then $x\gamma \equiv_K y\gamma$.
As γ is arbitrarily fixed, $x\gamma \equiv_K y\gamma \ \forall \ \gamma \in R \implies x \equiv_{(K:R)} y$, by Remark 2.
This implies $(K : R)$ is an e-prime strong ideal of S.

Proposition 3.7. Let S be a right permutable seminearring, R be a monogenic S-semigroup and K be a strong ideal of R such that $SR \not\subseteq K$. If $(K : R)$ is an e-prime strong ideal of S, then K is an e-prime strong ideal of R.

Proof. Suppose that $(K : R)$ is an e-prime strong ideal of S and $a \in S$, $\gamma_1, \gamma_2 \in R$ and $as\gamma_1 \equiv_K as\gamma_2 \ \forall \ s \in S$.
We need to show that $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
Suppose $aR \not\subseteq K$ and $\gamma_1 \not\equiv_K \gamma_2$. If $aR \not\subseteq K$, then $a \notin (K : R)$.
Nevertheless, because R is monogenic, there exists $\gamma_0 \in R$ such that $S\gamma_0 = R$.
Hence $\gamma_1 = x\gamma_0$ and $\gamma_2 = y\gamma_0$ for some $x, y \in S$.
If $\gamma_1 \not\equiv_K \gamma_2$, then $x\gamma_0 \not\equiv_K y\gamma_0 \implies x \not\equiv_{(K:R)} y$.
Since $(K : R)$ is equiprime strong and $a \notin (K : R)$, $x \not\equiv_{(K:R)} y$.
There exists $m \in S$ such that $amx \not\equiv_{(K:R)} amy \implies amx \not\equiv_{(K:S\gamma_0)} amy$.
Then $amxs'\gamma_0 \not\equiv_K amys'\gamma_0$ for some $s' \in S$.
$\implies ams'x\gamma_0 \not\equiv_K ams'y\gamma_0$. (since right permutable)
$\implies ams'\gamma_1 \not\equiv_K ams'\gamma_2$.
Hence there exists $ms' \in S$ such that $a(ms')\gamma_1 \not\equiv_K a(ms')\gamma_2$.
A contradiction to our assumption $as\gamma_1 \equiv_K as\gamma_2 \ \forall \ s \in S$.
Hence $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
Therefore K is an e-prime strong ideal of R.

Proposition 3.8. If K is a strong ideal of S-semigroup R, then $(K : R)$ is a strong ideal of S.

Proof. Let $a, b \in (K : R)$. Then by definition of $(K : R)$ we get $aR \subseteq K$, $bR \subseteq K$.
This implies $a\gamma, b\gamma \in K \ \forall \ \gamma \in R$.
Then $a\gamma + b\gamma \in K \ \forall \ \gamma \in R$ (since K is an ideal)
$\implies (a + b)\gamma \in K \ \forall \ \gamma \in R$
$\implies a + b \in (K : R)$.
Take $z \in s + (K : R)$. Then $z = s + a$ for some $a \in (K : R)$.
Since $a \in (K : R)$, we have $aR \subseteq K \implies a\gamma \in K \ \forall \ \gamma \in R$
Let $\gamma \in R$ be fixed. Then $z\gamma = (s + a)\gamma = s\gamma + a\gamma = s\gamma + p_1$ for some $p_1 \in K$.
$0 + z\gamma = p_2 + s\gamma$ (since K is an ideal).
$\implies z\gamma \equiv_K s\gamma \implies z \equiv_{(K:R)} s \implies z \in (K : R) + s$.
Let $s_1 \equiv_{(K:R)} s_2$. Then $a_1 + s_1 = a_2 + s_2$ for some $a_1, a_2 \in (K : R)$.
Now, take $\gamma \in R$. Then $(a_1 + s_1)\gamma = (a_2 + s_2)\gamma$
$\implies a_1\gamma + s_1\gamma = a_2\gamma + s_2\gamma$.
Since $a_1\gamma, a_2\gamma \in K$, we get $s_1\gamma \equiv_K s_2\gamma$
As $\gamma \in R$ is arbitrary, we have $s_1\gamma \equiv_K s_2\gamma$, $\forall \ \gamma \in R$.
This implies $s_1 \equiv_{(K:R)} s_2$. Hence $s_1 + \in (K : R) + s_2$.
Let $s, s' \in S$ be such that $z \in s((p : R) + s')$.

$\implies z = s(a + s')$ for some $a \in (K : R)$.
Let $\gamma \in R$. Then $z\gamma = s(a + s')\gamma = s(a\gamma + s'\gamma) = s(p_1 + s'\gamma) \subseteq K + ss'\gamma$ (left ideal of K).
Hence $z\gamma = p_2 + (ss')\gamma \implies z\gamma \equiv_K (ss')\gamma \implies z \equiv_{(K:R)} ss' \implies z \in (K : R) + ss'$.
Let $z \in (K : R)S$. Then $z = as$ for some $a \in (K : R)$.
Now, take $\gamma \in R$. Then $z\gamma = as\gamma = a(s\gamma) = a\gamma_1$ $(\gamma_1 \in R)$
Since $a \in (K : R) \implies a\gamma \in K\ \forall\ \gamma \in R$.
$\implies z\gamma = a\gamma_1 \in K \implies z \in (K : R)$.
Therefore $(K : R)$ is a strong ideal of the seminearring S.

Proposition 3.9. If K is a c-prime strong ideal of R then $(K : R)$ is a c-prime strong ideal of S.

Proof. First, we assume that K is a c-prime strong ideal of R. That is, for $a \in S, \gamma \in R$ if $a\gamma \in K$ then $aR \subseteq K$ or $\gamma \in K$.
Let $x, y \in S$ be such that $xy \in (K : R)$. Suppose $y \in (K : R)$, then we are done.
If $y \notin (K : R)$ then $yR \not\subseteq K$. This implies $y\gamma \notin K$ for some $\gamma \in R$.
$xy \in (K : R) \implies xyR \subseteq K \implies xy\gamma \in K, \forall\ \gamma \in R$
$\implies x(y\gamma) \in K\ \forall\ \gamma \in R$
Since K is c-prime strong and $y\gamma \notin K \implies xR \subseteq K \implies x \in (K : R)$.
Hence $(K : R)$ is a c-prime strong ideal of S.

Proposition 3.10. If S is a right permutable seminearring, R is a monogenic S-semigroup and K is a strong ideal of R such that $SR \not\subseteq K$ and $S_d \setminus (K : R) \neq \emptyset$ then K is c-e-prime strong if and only if K is c-prime strong.

Proof. Suppose K is a c-prime strong ideal of S-semigroup R. Let $a \in S, \gamma_1, \gamma_2 \in R$ be such that $a\gamma_1 \equiv_K a\gamma_2$.
Then we have to show that $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
Since R is monogenic, there exists $\gamma_0 \in R$ such that $S\gamma_0 = R$. This implies $\gamma_1 = x\gamma_0, \gamma_2 = y\gamma_0$ for some $x, y \in S$.
Then $ax\gamma_0 \equiv_K ay\gamma_0$.
Since K is c-prime strong, $ax \equiv_{(K:R)} ay$ or $\gamma_0 \in K$.
If $\gamma_0 \in K$, then $R = S\gamma_0 \subseteq SK \subseteq K$, a contradiction to $SR \not\subseteq K$.
So $ax \equiv_{(K:R)} ay$. Since $(K : R)$ is an ideal of S, we get $s_d ax \equiv_{(K:R)} s_d ay$ for some $s_d \in S_d \setminus (K : R)$.
$\implies s_d xa \equiv_{(K:R)} s_d ya$. (since S is right permutable).
By Proposition 3.9, we get $xa \equiv_{(K:R)} ya$ (since $s_d \notin (K : R)$)
$\implies x \equiv_{(K:R)} y$ or $a \in (K : R)$.
If $x \equiv_{(K:R)} y$, then $x\gamma_0 \equiv_K y\gamma_0$. This implies $\gamma_1 \equiv_K \gamma_2$ and we are done.
If $a \in (K : R)$, then $aR \subseteq K$.
Thus K is c-e-prime strong.
Other way implication trivially holds. ie, K is c-e-prime strong $\implies$ K is c-prime.

Proposition 3.11. If S is a right permutable seminearring, R is a monogenic S-semigroup and K is a strong ideal of R together with $SR \not\subseteq K$, then K is e-prime strong if and only if K is c-e-prime strong.

Proof. Suppose that K is e-prime strong ideal of S-semigroup R.
Let $a \in S, \gamma_1, \gamma_2 \in R$ be such that $a\gamma_1 \equiv_K a\gamma_2$.
Then to prove K is c-e-prime, we need to show that either $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$.
Suppose $aR \nsubseteq K$ and $\gamma_1 \not\equiv_K \gamma_2$.
As R is monogenic, $S\gamma_0 = R$ for $\gamma_0 \in R \setminus K$.
(If $\gamma_0 \in K$, then $R = S\gamma_0 \subseteq SK \subseteq K \implies R = K$, a contradiction to $SR \nsubseteq K$.)
Then $\gamma_1 = x\gamma_0, \gamma_2 = y\gamma_0$ for some $x, y \in S$.
If $\gamma_1 \not\equiv_K \gamma_2$, then $x\gamma_0 \not\equiv_K y\gamma_0 \implies x \not\equiv_{(K:R)} y$.
If K is e-prime strong ideal of S, then $(K : R)$ is e-prime strong ideal of S.
$aR \nsubseteq K \implies a \notin (K : R)$ and we have $x \not\equiv_{(K:R)} y$. This implies $amx \not\equiv_{(K:R)} amy$ for some $m \in S$.
$\implies axm \not\equiv_{(K:R)} aym$ (since S is right permutable).
Since $(K : R)$ is a strong ideal of S, $ax \not\equiv_{(K:R)} ay$
$\implies ax\gamma_0 \not\equiv_K ay\gamma_0 \implies a\gamma_1 \not\equiv_K a\gamma_2$, a contradiction to our assumption.
Hence $aR \subseteq K$ or $\gamma_1 \equiv_K \gamma_2$. Thus K is c-e-prime strong.
Converse follows trivially.

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References

1. Bhavanari, S., Kuncham, S.P.: *Near rings, fuzzy ideals, and graph theory.* CRC press New York, USA, 2013.
2. Booth, G.L., Groenewald, N.J., Veldsman, S.: A Kurosh-Amitsur prime radical for near-rings, *Commun. Algebra.* **18** (1990) 3111–3122. https://doi.org/10.1080/00927879008824063
3. Booth, G.L., Groenewald, N.J.: Equiprime left ideals and equiprime N-groups of a near-ring, *Beitr Algebra Geom.* **8** (1992) 25–38.
4. Groenewald, N.J.: Different prime ideals in near-rings, *Commun. Algebra.* **19** (1991) 2667–2675. https://doi.org/10.1080/00927879108824287
5. Holcombe, W.M.: Primitive near-rings, *PhD Thesis*, University of Leeds, England, 1970.
6. Juglal, S., Groenewald, N.J., Lee, K.S.E.: Different prime R-ideals, *Algebra Colloq.* **17** (2010) 887–904. https://doi.org/10.1142/S1005386710000830
7. Koppula, K., Kedukodi, B.S., Kuncham, S.P.: Congruences in seminearrings and their correspondence with strong ideals. *Alg. Struc. Appl.* In Press (2023).
8. Koppula, K., Kedukodi, B.S., Kuncham, S.P.: On prime strong ideals of a seminearring, *Mat. Vesnik* **72** (2020) 243–256.
9. Pilz, G.: *Near-rings: the theory and its applications*, Revised edition, Elsevier, 1983.
10. Prakash, P., Kedukodi, B.S., Kuncham, S.P., Koppula, K.: Isomorphism theorems for S-semigroups with respect to strong ideal, *Communicated*, (2023).
11. Prakash, P., Kedukodi, B.S., Kuncham, S.P., Koppula, K.: Prime strong ideals of S-semigroup, *Communicated*, (2024).
12. Taşdemir, F., Atagün, A.O.: Equiprime N-ideals of Monogenic N-groups, *Hacet. J. Math. Stat.* **40** (2011) 375–382.
13. Veldsman, S.: On equiprime near-rings, *Commun. Algebra* **20** (1992) 2569–2587. https://doi.org/10.1080/00927879208824479
14. VanHoorn, W.G., VanRootselaar, B.: Fundamental notions in the theory of seminearrings, *Composit. Math.* **18** (1967) 65–78. http://www.numdam.org/item?id=CM_1967__18_1-2_65_0

Prakash Padoor: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,Manipal, 576104, Karnataka, India
Email address: prakash.padoor@gmail.com

Babushri Srinivas Kedukodi: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,Manipal, 576104, Karnataka, India
Email address: babushrisrinivas.k@manipal.edu

Syam Prasad Kuncham: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,Manipal, 576104, Karnataka, India
Email address: syamprasad.k@manipal.edu

Kavitha Koppula: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,Manipal, 576104, Karnataka, India
Email address: kavitha.koppula@manipal.edu