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FRACTIONAL STOCHASTIC HEAT CONDUCTION EQUATION OF HYPERBOLIC TYPE

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Abstract. We consider the initial problem for one class of fractional stochastic differential equations with white noise in time and space for solutions of distribution type. Existence and singularity theorems for solutions are established and explicit representations for them are found.

1. Introduction

The well known classical model of heat conduction is based (see e.g. [1]) on Fourier's law

$$q(t,x) = -a\nabla T(t,x), t \geq 0, x \in \mathbb{R}^d, \tag{1}$$

where q-heat flux, T-temperature, ∇-gradient operator, a-heat transfer coefficient. In the case when there are no heat sources, the law of conservation of energy has the following form

$$\rho c_\rho \frac{\partial T(t,x)}{\partial t} = -divq(t,x), \tag{1.2}$$

where t-time, ρ-density, c_ρ- specific heat conductivity of the medium body.

Substituting (1.2) into (1.1) we obtain the classical heat conduction equation [1,2-4] in the form

$$\frac{\partial T(t,x)}{\partial t} = D\triangle T(t,x), \triangle T = \sum_{j=1}^{d} \frac{\partial^2 T}{\partial x^2}, \tag{1.3}$$

where $D = \frac{a}{\rho c_\rho}$-coefficient of thermal conductivity, $\triangle$- Laplace operator in $\mathbb{R}^d$.

The heat conduction equation (1.3) is a parabolic type equation and has a paradoxical "unphysical" property — infinite velocity of propagation of disturbances, which indicates a very narrow field of application of Fourier's law (1.1). A natural question arises as to how to "correct" the Fourier law so that the corresponding model shows a finite perturbation propagation velocity. The question formulated here was answered in the early 1960s by French mathematicians Cattaneo C. and Vernotte P. in [5,6].

They proposed to use instead of (1.1) another differential model in the form of

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$$q(t,x) = -a\nabla T(t,x) - \tau \frac{\partial q(t,x)}{\partial t}, \tag{1.4}$$

where τ-relaxation or delay time. As can be seen, the model (1.4) differs from the Fourier law (1.1) by the presence of an additional nonstationary term proportional to the value of τ. At $\tau \neq 0$ (1.4) we obtain the hyperbolic heat conduction equation, namely

$$\tau \frac{\partial^2 T(t,x)}{\partial t^2} + \frac{\partial T(t,x)}{\partial t} = d\triangle T(t,x). \tag{1.5}$$

The model (1.5) already demonstrates a finite speed of perturbation propagation and is widely used to solve various problems in the theory and practice of heat and mass transfer [7-10]. Note that in mathematical physics equations of the form (1.5) are called the telegraph equation [11]. In essence, the Cattaneo-Vernotte model is a differential-difference analogue of the Fourier law and has the form (see, e.g., [12]):

$$q(t,x)\Big|_{t+\tau} = -a\triangle(t,x), \tag{1.6}$$

where the left part of the equation is calculated at $t+\tau$ with shift. At $\tau = 0$, the differential difference relation (1.6) transforms into the Fourier law (1.1). If we formally decompose the left part of (1.6) into a series over τ and keep only two main terms of the expansion, we obtain the Cattaneo-Vernotte model.

Here we note that in recent decades there has been a rapid growth of the theory of stochastic partial differential equations with fractional orders of derivatives both in time and in spatial coordinates. This is primarily due to a wide range of applications in physics, engineering, in such areas as electromagnetism, fluid mechanics, signal propagation, hydrology, fractional kinetics, electrochemistry, viscoelasticity, optics, robotics, biomedicine, etc. [13-16]. Many publications are devoted to fractional heat conduction equations of parabolic type. Very few works refer to heat conduction equations of hyperbolic type. Stochastic partial derivative equations of fractional order in time are considered in [17].

This paper considers a fractional stochastic equation of hyperbolic type, namely an equation

$$\tau \frac{\partial^{2\alpha} u(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}} = D\triangle u(t,x) + \sigma W(t,x), t \geq 0, x \in \mathbb{R}^2, \tag{1.7}$$

where τ-relaxation time, $\frac{\partial^\alpha}{\partial t^\alpha}$-derivative Gerasimov-Caputo of order $\alpha \in (0,1], D > 0, \sigma \in \mathbb{R}$-defined constants, $\triangle u$-operator of Laplace,

$$W(t,x) = W(t,x,\omega) = \frac{\partial}{\partial t}\frac{\partial^2 B(t,x_1,...,x_d)}{\partial x_1...\partial x_d} \tag{1.8}$$

is white noise in time and space, and

$$B(t,x) = B(t,x,\omega), t \geq 0, x \in \mathbb{R}^d, \omega \in \Omega$$

brownian sheet with the probability law $\mathbb{P}$.

The boundary conditions have the form

$$u(0,x) = \delta_0(x) \text{ point mass in } 0 \tag{1.9}$$

and

$$\lim_{x\to\pm\infty} u(t,x) = 0. \tag{1.10}$$

The fractional derivative in stochastic media is more useful in comparison with the integer-order derivatives in many situations, for example, in the analysis of propagation of signals and waves, including ocean waves around oil platforms or in the study of ultrasound properties in solids. In particular, the fractional heat conduction equation in stochastic media of the form (1.7) can be used in describing anomalous diffusion (subdiffusion or superdiffusion) of heat flux propagation. It is connected with the power conservation law [18].

This paper consists of an introduction, seven sections and a conclusion. Section 2-6 gives preliminary information and some auxiliary results. Section 7 is devoted to analyzing the properties of the Fourier-Laplace transform. Section 8 establishes the uniqueness theorem for the solution of problem (1.7)-(1.10) and derives an explicit representation of the solution.

2. The space of temporary distributions

For the convenience of the reader, let us recall some basic properties of the Schwarz space S — of rapidly decreasing smooth functions (the space of basic functions) and its dual space S' — of temporary increasing distributions.

Let n be a given natural number. Let $S = S(\mathbb{R}^n)$ be the space of rapidly decreasing smooth functions f on $\mathbb{R}^n$ equipped with a family of seminorms of the form

$$\|f\|_{k,\alpha} = \sup_{y\in\mathbb{R}^n}\{(1+|y|^k)|\partial^\alpha f(y)|\} < \infty,$$

where $k \in \mathbb{N}\cup\{0\}, \alpha = (\alpha_1, ..., \alpha_n)$ multi-index with $\alpha_j = 0, 1, ..., j = 1, ..., m$ and

$$\partial^\alpha f = \frac{\partial^{|\alpha|}}{\partial y_1^{\alpha_1} ... \partial y_n^{\alpha_n}} f$$

for $|\alpha| = \alpha_1 + ... + \alpha_n$.

Under the above conditions, S will be a Fréchet space. Let $S' = S'(\mathbb{R}^n)$ be a dual space for S, the so-called space of temporary growing distributions. We denote by $\mathcal{B}$ the family of all Borel subsets of $S' = S'(\mathbb{R}^n)$ equipped with weak $*$ topology. If $\Phi \in S'$ and $f \in S$ then via

$$\Phi(f) \text{ or } <\Phi, f> \tag{2.1}$$

denote the action of Φ on f.

Example 2.1. For** $y \in \mathbb{R}$**, define the delta function** δ_y **on** $S(\mathbb{R})$ **by the following equations

$$\delta_y(\varphi) = \varphi(y).$$

Then** δ_y **will be a temporary increasing distribution.

Example 2.2. Consider the function D, $\varphi \in S(\mathbb{R})$ ***by***

$$D[\varphi] = \varphi'(y).$$

Then the generalized derivative of D ***will be a temporary increasing distribution.***

Example 2.3. Let T ***be a temporary distribution, i.e.,*** $T \in S'(\mathbb{R})$***. Let us define the derivative of*** T' ***from*** T ***by***

$$T'(\varphi) = -[\varphi'], \varphi \in S.$$

Then T' ***will again be a temporary distribution.***

3. Some properties of Mittag-Leffler functions

We denote the two-parametric Mittag-Leffler function by $E_{\alpha,\beta}(z)$, which is defined by the equations

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{3.1}$$

where $z, \alpha, \beta \in \mathbb{C}, Re(\beta) > 0$ and Γ is the Euler gamma function.

The one-parameter Mittag-Leffler function is denoted by $E_\alpha(z)$, which is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{3.2}$$

where $z, \alpha \in \mathbb{C}, Re(\alpha) > 0$.

Remark 3.3. Note that $E_\alpha(z) = E_{\alpha,1}(z)$ ***and such that***

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{3.3}$$

4. Gerasimov-Kaputo fractional derivative

In this section, we give definitions and some properties of the fractional Gerasimov-Caputo derivative and its Laplace transform.

We denote $g_\alpha(t) = t^{\alpha-1}\Gamma(\alpha)$ at $\alpha > 0, t > 0$, J_t^0-an identical operator and

$$J_t^\alpha h(t) = (g_\alpha * h)(t) = \int_0^t g_\alpha(t-s)h(s)hs. \tag{4.1}$$

Let $m - 1 < \alpha \leq m \in \mathbb{N}, D_t^m$-ordinary derivative of order m. Then the relation

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}(h(t) - \sum_{k=0}^{m-1} h^k(0) g_{k+1}(t)) \tag{4.2}$$

defines the fractional Gerasimov-Caputo derivative of order α of the function $h(t)$.

If the function h is nonsmooth, these derivatives are interpreted in the sense of distributions.

Example 4.1. ***If*** $h(t) = t$ ***and*** $\alpha \in (0,1)$ ***then***

$$D^{\alpha}h(t) = \frac{t1-\alpha}{(-\alpha)\Gamma(1-\alpha)}. \tag{4.3}$$

In particular, choosing $\alpha = 1/2$ ***we obtain***

$$D^{1/2}h(t) = \frac{2\sqrt{t}}{\sqrt{\pi}}. \tag{4.4}$$

Recall that the Laplace transform $\mathcal{L}$ is defined as equations of the form

$$\mathcal{L}h(\lambda) = \int_0^{\infty} e^{-\lambda t}h(t)dt \tag{4.5}$$

for all h for which the integral in (4.5) converges.

We give some properties of the Laplace transform, which will be needed in the future:

$$\mathcal{L}[\frac{\partial^{\alpha}h(t)}{\partial t^{\alpha}}](\lambda) = \lambda^{\alpha}(\mathcal{L}h)(\lambda) - \lambda^{\alpha-1}h(0), \tag{4.6}$$

$$\mathcal{L}[\frac{\partial^{2\alpha}h(t)}{\partial t^{\alpha}}](\lambda) = \lambda^{2\alpha}(\mathcal{L}h)(\lambda) - \lambda^{2\alpha-1}h(0), \tag{4.7}$$

$$\mathcal{L}[E_{\alpha}(bt^{\alpha})](\lambda) = \frac{\lambda^{\alpha-1}}{\lambda^{\alpha}-b}, \tag{4.8}$$

$$\mathcal{L}[E_{\alpha}(bt^{2\alpha})](\lambda) = \frac{\lambda^{2\alpha-1}}{\lambda^{2\alpha}-b}, \tag{4.9}$$

$$\mathcal{L}[t^{\alpha-1}E_{\alpha,\alpha}(-bt^{\alpha})](\lambda) = \frac{1}{\lambda^{\alpha}+b}, \tag{4.10}$$

$$\mathcal{L}[t^{2\alpha-1}E_{2\alpha,2\alpha}(-bt^{2\alpha})](\lambda) = \frac{1}{\lambda^{2\alpha}+b}. \tag{4.11}$$

The convolution theorem for the Laplace transform states that

$$\mathcal{L}(\int_0^t f(t-r)g(\tau)dr)(\lambda) = \mathcal{L}f(\lambda)\mathcal{L}g(\lambda)$$

or

$$\int_0^t f(t-\omega)g(\omega) = \mathcal{L}^{-1}(\mathcal{L}f(\lambda)\mathcal{L}g(\lambda))(t). \tag{4.12}$$

5. White noise in time-space

Let n be a fixed natural number and let $n = 1 + d$, where d is the dimension of $\mathbb{R}^{\alpha}$. Let us define $\Omega = S'(\mathbb{R}^n)$ equipped with a weak $*$ topology. This space will be the base of our basic probability space in the sense that the probability measure $\mathbb{P}$ will be defined via the following theorem:

Theorem 5.1 (Bochner-Milnos Theorem). There exists a single probability measure $\mathbb{P}$ ***on*** $\mathcal{B}(S'(\mathbb{R}^n))$ ***with the following property***

$$\mathbb{E}[e^{i(\cdot,\varphi)}] \int\limits_{S'} e^{i(\omega,\varphi)} d\mu(\omega) = e^{-1/2\|\varphi\|^2}, i = \sqrt{-1}$$

or all $\varphi \in S(\mathbb{R}^n)$***, where*** $\|\varphi\|^2 = \|\varphi\|^2_{2(\mathbb{R}^n)}$***,*** $(\omega, \varphi) = \omega(\varphi)$ ***is an action*** $\omega \in S'(\mathbb{R}^n)$ ***on*** $\varphi \in S(\mathbb{R}^n)$ ***and through*** $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ ***denotes the mathematical expectation with respect to*** $\mathbb{P}$***.***

We will say that the triple $(S'(\mathbb{R}^n), \mathcal{B}(S'(\mathbb{R}^n), \mathbb{P})$ is called the probability space of white noise, and $\mathbb{P}$ the probability measure of white noise.

The measure $\mathbb{P}$ is sometimes also called the normal Gaussian measure on $S'(\mathbb{R}^n)$. It is easy to prove that if $\varphi \in L^2(\mathbb{R}^n)$ and there is a set $\varphi_k \in S(\mathbb{R}^n)$ such that $\varphi_n \to \varphi$ in $L^2(\mathbb{R}^n)$, then

$$< \omega, \varphi >= \lim_{k\to\infty} < \omega, \varphi_k > \text{ exists in } L^2(\mathbb{R}^n)$$

and it does not depend on the choice of $\{\varphi_k\}$. In particular, if we define

$$\widetilde{B}(x) = \widetilde{B}(x_1, ..., x_n, \omega) =< \omega, \mathcal{X}_{[0,1]\times\cdots\times[0,x_n]} >, x = (x_1, ..., x_n) \in \mathbb{R}^n$$

where $[0, x_i]$ is interpreted as $[x_i, 0]$ if $x_i < 0$, then $\widetilde{B}(x, \omega)$ has a continuous version $B(x, \omega)$, which becomes an n-parametric Brownian motion in the following sense:

By n-parametric Brownian motion we mean a family $\{B(x, \cdot)\}_{\in\mathbb{R}^n}$ of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

1) $B(0, \cdot) = 0$ almost certainly in relation to $\mathbb{P}$;
2) $\{B(x, \omega)\}$ is a continuous Gaussian stochastic process;
3) For all $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n_+, B(x, \cdot), B(y, \cdot)$ have covariance $\prod_{i=1}^n x_i \wedge y_i$. For the general case $x, y \in \mathbb{R}^n$, the covariance will be

$$\prod_{i=1}^n \int\limits_{\mathbb{R}} \Theta_{x_i}(s)\Theta_{y_i}(s) ds,$$

where $\Theta_x(t_1, ..., t_n) = \Theta_{x_1}(t_1), ..., \Theta_{x_n}(t_n)$, with

$$\Theta_{x_j} = \begin{cases} 1, & \text{if } 0 < S < x_j \\ -1, & \text{if } x_j < S \leq 0 \\ 0, & \text{in other cases.} \end{cases}$$

It can be proved that the process $\widetilde{B}(x, \omega)$ defined above has a modification $B(x, \omega)$ which satisfies all these properties. This process $B(x, \omega)$ then becomes an n-parameter Brownian motion. Note that for $n = 1$ we obtain the classical

1-parameter Brownian motion $B(t)$ if we restrict ourselves to the case $t \geq 0$. For $n \geq 2$ we obtain the so-called Brownian sheet.

With this definition of Brownian motion the n-parametric Wiener-Ito integral is naturally called, i.e., the following integral from $\varphi \in L^2(\mathbb{R}_n)$

$$\int\limits_{\mathbb{R}_n} \varphi(x) dB(x,\omega) = (\omega, \varphi), \omega \in S'(\mathbb{R}n).$$

It is easy to see that using the Bochner-Minlos theorem we can find a construction of n-parametric motion for any n. Moreover, we obtain a representation of the space Ω as the Fréchet space $S'(\mathbb{R}^n)$. This leads to success in many situations, such as the construction of the Hida-Mallaven derivative, which can be viewed as a stochastic gradient on Ω.

Next, let $n = 1 + d$ and let through

$$B(t,x) = B(t,x,\omega), t \geq 0, x \in \mathbb{R}n, \omega \in \Omega$$

denotes the time-space Brownian motion (also called Brownian sheet) with probabilistic law $\mathbb{P}$. Since this process is (t,x) continuous almost surely, we can define for almost all $\omega \in \Omega$ its derivatives with respect to t and x in the sense of distributions. Thus, we define time-space white noise to be $W(t,x) = W(t,x,\omega)$ through

$$W(t,x) = \frac{\partial}{\partial t} \frac{\partial^d B(t,x)}{\partial x_1 ... \partial x_d}. \tag{5.1}$$

In particular, for $d = 1$ and $x_1 = t$ we obtain the well-known identity

$$W(t) = \frac{d}{dt} B(t) \text{ in } S'.$$

Definition (5.1) can also be interpreted as an element of the Hida space $(S)^*$ of stochastic distributions (see [15]). In particular, it follows from [15] that the Ito-Skorokhod integral with respect to $B(dt, dx)$ can be expressed as

$$\int\limits_0^T \int\limits_{\mathbb{R}^d} f(t,x,\omega) B(dt,dx) = \int\limits_0^T \int\limits_{\mathbb{R}^d} f(t,x,\omega) \lozenge W(t,x) dt dx \tag{5.2}$$

where the symbol $\lozenge$ denotes the Wick product. In particular, if $f(t,x,\omega) = f(t,x,)$ is deterministic, then

$$\int\limits_0^T \int\limits_{\mathbb{R}^d} f(t,x,) B(dt,dx) = \int\limits_0^T \int\limits_{\mathbb{R}^d} f(t,x,) W(t,x) dt dx. \tag{5.3}$$

6. Fourier transform of the solution at $d = 1$

In order to find a solution to the one-dimensional x equation

$$\tau \frac{\partial^{2\alpha} u(t,x)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(t,x)}{\partial t^{\alpha}} = d^2 \frac{\partial^2 u(t,x)}{\partial x^2} + \tau W(t,x), \leq \alpha \leq 1 \tag{6.1}$$

with initial conditions

$$\begin{cases} u(0,x) = \delta(x) \\ u_t(0,x) = 0, \qquad \text{for } 1/2 < \alpha \leq 1 \end{cases} \tag{6.2}$$

and

$$u(0,x) = \delta(x) \text{ for } 0 < \alpha \leq 1/2 \tag{6.3}$$

consider the Fourier transform

$$U(t,\beta) = \int_{-\infty}^{\infty} e^{i\beta x} u(t,x)dx, \tag{6.4}$$

which satisfies the equation

$$\tau \frac{\partial^{2\alpha} U}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} U}{\partial t^{\alpha}} + d^2\beta^2 U = \int_{-\infty}^{\infty} e^{i\beta x} W(t,x)dx, \tag{6.5}$$

with initial conditions

$$\begin{cases} u(0,\beta) = 1 \\ u_t(0,\beta) = 0, \quad \text{for } 1/2 < \alpha \leq 1 \end{cases} \tag{6.6}$$

and

$$u(0,\beta) = 1 \text{ for } 0 < \alpha \leq 1/2 \tag{6.7}$$

Integration (6.5) with conditions (6.6) or (6.7) can be carried out through Laplace transform implementations of the form

$$\mathcal{L}U(t,\beta)(\lambda) = \int_{0}^{\infty} e^{\lambda t} U(t,\beta)dt. \tag{6.8}$$

In this paper we use fractional Gerasimov-Kaputo derivatives, which allow us to consider initial conditions in terms of integer order derivatives. This allows us to obtain the following the formula for the Laplace transform of derivatives of order α, namely

$$\mathcal{L}\, {}^cD_t^{\alpha} U(t,\beta)(\lambda) = \lambda^{\alpha} \mathcal{L}U(t,\beta)(\lambda) - \sum_{k=1}^{m-1} \lambda^{\alpha-1-k} D^k U(t,\beta)\big|_{t=0} \tag{6.9}$$

where $m - 1 = \lfloor \alpha \rfloor$.

Formula (6.9) is established as follows:

$$\mathcal{L}\, {}^cD_t^{\alpha} U(t,\beta)(\lambda) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{\infty} e^{\lambda t} \left\{ \int_{0}^{t} \frac{\frac{\partial^m}{\partial t^m} U(z,\beta)}{(t-z)^{1+\alpha-m}} dz \right\} dt =$$

$$= \frac{1}{\Gamma(m-\alpha)} \int\limits_0^\infty \frac{\partial^m}{\partial z^m} U(z,\beta) \Big\{ \int\limits_z^\infty \frac{e^{-\lambda t}}{(t-z)^{1+\alpha-m}} dt \Big\} dz =$$

$$= \lambda^{\alpha-m} \int\limits_0^\infty e^{\lambda z} \frac{\partial^m U(z,\beta)}{\partial z^m} dz.$$

Let us now use the well-known formula for the Laplace transform of integer derivatives in the form

$$\int\limits_0^\infty e^{\lambda z} \frac{\partial^m}{\partial z^m} U(z,\beta) dz = \lambda^m \mathcal{L}U(t,\beta)(\lambda) - \sum_{k=1}^{m-1} \lambda^{m-1-k} \frac{\partial^k U(t,\beta)}{\partial t^k}\Big|_{t=0}$$

and we obtain (6.9). It follows from (6.9) that since $U_t(0,\beta) = 0$, there is no difference between the cases $0 < \alpha \leq 1/2$ and $1/2 < \alpha \leq 1$.

After some calculations on the basis of (6.9), we will see that the Laplace transform of the solution of equation (6.5) taking into account the initial conditions leads to the formula

$$\mathcal{L}U_\alpha(t,\beta)(\lambda) = \frac{\lambda^{2\alpha-1} + \lambda^{\alpha-1}}{\tau\lambda^{2\alpha} + \lambda^\alpha + d^2\beta^2} = H(\lambda,\beta) \tag{6.10}$$

for all $0 < \alpha \leq 1$.

The problem of finding the inverse Laplace transform for the homogeneous equation (6.10) at $\sigma = 0$ is solved in the following theorem.

Theorem 6.1 The Fourier transform of solutions of problems (6.1), (6.2) and (6.1), (6.3) can be written in the following equivalent forms:

$$U_\alpha(t,\beta) = E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(1+\eta_2)t^\alpha}{\eta_1 - \eta_2}[\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)] =$$

$$+\frac{1}{2}\Big[(1 + \frac{1}{\sqrt{1-4\tau d^2\beta^2}})E_{\alpha,1}(\eta_1 t^\alpha) +$$

$$+(1 - \frac{1}{\sqrt{1-4\tau d^2\beta^2}})E_{\alpha,1}(\eta_1 t^\alpha)\Big], t > 0, \tag{6.11}$$

where

$$\eta_1 = \frac{1}{2\tau}\Big(-1 + \frac{1}{\sqrt{1-4\tau d^2\beta^2}}\Big), \eta_2 = \frac{1}{2\tau}\Big(-1 - \frac{1}{\sqrt{1-4\tau d^2\beta^2}}\Big).$$

Proof. It will be convenient to write (6.10) as follows

$$\mathcal{L}U_\alpha(t,\beta)(\lambda) = \lambda \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \eta_1} \cdot \frac{1}{\lambda^\alpha - \eta_2}. \tag{6.12}$$

Then we apply the relations

$$\int\limits_0^\infty e^{-\lambda t} E_{\alpha,1}(\eta_j t^\alpha) dt = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \eta_j}, j = 1,2 \tag{6.13}$$

and

$$\int_0^\infty e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha}(\eta_j t^\alpha) dt_{1/2} = \frac{1}{\lambda^\alpha - \eta_j}, j = 1, 2 \tag{6.14}$$

valid for $\lambda > \eta_j$.

It is easy to check the relation (6.13), i.e.

$$\int_0^\infty e^{-\lambda t} E_{\alpha,1}(\eta_j t^\alpha) dt = \sum_{k=0}^\infty \frac{\eta_j^k}{\Gamma(\alpha k+1)} \int_0^\infty e^{-\lambda t} t^{\alpha k} dt =$$

$$= \frac{1}{\lambda} \sum_{k=0}^\infty \left(\frac{\eta_j}{\lambda^\alpha}\right)^k = \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \eta_j}.$$

From the last step it will be clear why formulas (6.13) and (6.14) are true when $\lambda > \eta_j^{1/\alpha}$.

To reverse the first term in (6.12) we write

$$\lambda \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \eta_1} \cdot \frac{\lambda^{\alpha-1}}{\lambda^\alpha - \eta_2} = \int_0^\infty \lambda e^{-\lambda t} \left[\int_0^t E_{\alpha,1}(\eta_1 z) E_{\alpha,1}(\eta_2 (t-z)^\alpha) dz \right] dt =$$

$$= -e^{-\lambda t} \int_0^t E_{\alpha,1}(\eta_1 z^\alpha) E_{\alpha,1}(\eta_2 (t-z)^\alpha) dz \Big|_{t=0}^{t=\infty} + \int_0^\infty \lambda e^{-\lambda t} E_{\alpha,1}(\eta_1 t^\alpha) dt +$$

$$+ \int_0^\infty \lambda e^{-\lambda t} dt \cdot \int_0^t E_{\alpha,1}(\eta_1 z^\alpha) \frac{d}{dt} E_{\alpha,1}(\eta_2 (t-z)^\alpha) dz \tag{6.15}$$

To prove the equality to zero of the second summand in (6.15) it is necessary to take into account the following formula [16]:

$$\int_0^t z^{\gamma-1} E_{\alpha,\gamma}(y z^\alpha)(t-z)^{\beta-1} E_{\alpha,\beta}(\omega (t-z)^\alpha) dz =$$

$$\frac{t^{\beta+\gamma-1}}{y-\omega} \sum_{k=0}^\infty \frac{t^{\alpha k}(y^{k+1} - \omega^{k+2})}{\Gamma(\alpha k + \beta + \gamma)} =$$

$$\frac{t^{\beta+\gamma-1}}{y-\omega} \left[y E_{\alpha,\beta+\gamma}(y t^\alpha) - \omega E_{\alpha,\beta+\gamma}(\omega t^\alpha) \right], \tag{6.16}$$

for $\beta, \gamma.0$ and $y \neq \omega$. Formula (6.16) for $\gamma = \beta = 1, y = r, \omega = \eta_2$ yields

$$\int_0^t E_{\alpha,1}(\eta_1 t^\alpha) E_{\alpha,1}(\eta_2 (t-z)^\alpha) dz =$$

$$\frac{t}{\eta_1 - \eta_2}\left[\eta_1 E\alpha, 2(\eta_1 t^\alpha) - \eta_2 E\alpha, 2(\eta_1 t^\alpha)\right].$$

By considering the asymptotic behavior of the Mittag-Leffler function directly it can be shown that the second term in (6.16) is correct.

Since

$$\frac{d}{dt}E_{\alpha,1}(\eta_2(t-z)\alpha) = \sum_{k=0}^{\infty}\frac{(\eta_2(t-z)^\alpha)^{k-1}}{\Gamma(\alpha k+1)}\alpha k\eta_2(t-z)^{\alpha-1} =$$

$$= \sum_{k=1}^{\infty}\frac{(\eta_2(t-z)^\alpha)^{k-1}}{\Gamma(\alpha k)}\eta_2(t-z)^{\alpha-1} =$$

$$= \sum_{k=1}^{\infty}\frac{(\eta_2(t-z)^\alpha)^{k}}{\Gamma(\alpha k+\alpha)}\eta_2(t-z)^{\alpha-1} =$$

$$\eta_2(t-z)^{\alpha-1}E_{\alpha,\alpha}(\eta_2(t-z)^\alpha),$$

then we have

$$\lambda\frac{\lambda^{\alpha-1}}{\lambda^\alpha-\eta_1}\cdot\frac{\lambda^{\alpha-1}}{\lambda^\alpha-\eta_2} = \int_0^\infty e^{-\lambda t}E_{\alpha,1}(\eta_1 t^\alpha)dt + \eta_2\int_0^\infty e^{-\lambda t}\times$$

$$\times(t-z)^{\alpha-1}E_{\alpha,1}(\eta_1 t^\alpha)E_{\alpha,\alpha}(\eta_2(t-z)^\alpha)dz. \tag{6.17}$$

Hence the inverse Laplace transform has the form

$$U_\alpha(t,\beta) = E_{\alpha,1}(\eta_1 t^\alpha) + (1+\eta_2)\int_0^t(t-z)\alpha - 1\times$$

$$\times E_{\alpha,1}(\eta_1 z^\alpha)E_{\alpha,\alpha}(\eta_2(t-z)^\alpha)dz =$$

$$= E_{\alpha,1}(\eta_1 t^\alpha) + \frac{(1+\eta_2)}{\eta_1-\eta_2}\left[\eta_1 E_{\alpha,\alpha+1}(\eta_1 t^\alpha) - \eta_2 E_{\alpha,\alpha+1}(\eta_2 t^\alpha)\right]. \tag{6.18}$$

7. Explicit solutions for $\alpha = 1/2$ of the heat conduction equation of hyperbolic type

Let us consider the case $\alpha = 1/2$ for which it is possible to find a solution of equation (1.7) with initial conditions (1.10). For this purpose, let us write out the Fourier transform (6.4) in a more convenient form.

Theorem 7.1. ***For*** $\alpha = 1/2$ ***there is the following Fourier transform of the solution***

$$U_{1/2}(t,\beta) = \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{z^2}{4t}-z}\left\{\frac{e^{z\sqrt{1-4\tau d^2\beta 2}} - e^{-z\sqrt{1-4\tau d^2\beta 2}}}{\sqrt{1-4\tau\omega^2\beta 2}}\right\}dz+$$

$$+\frac{1}{2\sqrt{\pi t}}\int\limits_0^\infty e^{\frac{-z^2}{4t}-z}\left\{e^{z\sqrt{1-4\tau d^2\beta 2}}+e^{-z\sqrt{1-4\tau d^2\beta 2}}\right\}dz. \tag{7.1}$$

Proof. To go from the first representation of formula (6.18) to formula (7.1) we need to express explicitly the Mittag-Leffler functions $E_{1/2,1/2}(x)$ and $E_{1/2,1}(x)$.

We've got

$$E_{1/2,1/2}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma\left(\frac{k+3}{2}\right))}=\sum_{k=0}^{\infty}\frac{x^k}{\frac{k+1}{2}\Gamma\left(\frac{k+3}{2}\right))}=$$

$$\sum_{k=0}^{\infty}\frac{x^k2^{k+1}}{\Gamma\left(\frac{k}{2}+1\right))}=\sum_{k=0}^{\infty}x^k2^{k+1}F(\frac{k}{2}+1)\int_0^\infty e^{-\omega}\omega^{k/2}d\omega=$$

$$=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\omega^2}(e^{2\omega x}-1)d\omega. \tag{7.2}$$

Similarly, we find

$$E_{1/2,1}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma\left(\frac{k}{2}+1\right))}=\sum_{k=0}^{\infty}\frac{x^k2^k\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}k!}=$$

$$\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\omega^2+2x\omega}d\omega. \tag{7.3}$$

From formulas (7.2) and (7.3) we obtain

$$U_{1/2}(t,\beta)=\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-\omega^2-\frac{\omega}{\tau}\sqrt{t}+2\sqrt{t}\omega\sqrt{1-4\tau d^2\beta^2}}+$$

$$+\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\omega^2-\frac{\omega}{\tau}\sqrt{t}}\left\{\frac{e^{2\omega\sqrt{t}\sqrt{1-4\tau d^2\beta^2}}-e^{-2\omega\sqrt{t}\sqrt{1-4\tau d^2\beta^2}}}{\sqrt{1-4\tau d^2\beta^2}}\right\}d\omega+$$

$$+\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\omega^2-\frac{\omega}{\tau}\sqrt{t}}\left\{e^{2\omega\sqrt{t}\sqrt{1-4\tau d^2\beta^2}}-e^{-2\omega\sqrt{t}\sqrt{1-4\tau d^2\beta^2}}\right\}d\omega,$$

which corresponds to formula (7.1) after introducing the substitution of variables $2\sqrt{t}\omega=z$.

Theorem 7.2. The distribution obtained using the Fourier transform inversion (7.1) has the following form

$$U_{1/2}(t,x)=\frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{\omega^2}{4t}-\omega}\left\{I_0\left(\frac{1}{d}\sqrt{{}^2\omega^2-x^2}\right)+\right.$$

$$+\frac{\partial}{\partial\omega}I_0\left(\frac{1}{d}\sqrt{d\omega^2-x^2}1_{|x|<d\omega}+\right.$$

$$\left.+c[\delta(x-d\omega)+\delta(x+d\omega)]\right\}d\omega.$$

8. Explicit solution of the fractional stochastic heat conduction equation of hyperbolic type at $\alpha = 1/2$

Theorem 8.1. ***The feasible solution*** $u(t,x) \in S'$ ***of the fractional stochastic heat conduction equation of hyperbolic type (1.7)-(1.10) at*** $\alpha = 1/2$ ***is given by the formula***

$$u(t,x) = I_1 + I_2, \tag{8.1}$$

where

$$I_1 = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} \sum_{k=0}^{\infty} \frac{(-D(z\tau + \sqrt{t})|y|^2)^k}{\Gamma(\frac{k+2}{2})} \tag{8.2}$$

and

$$I_2 = \sigma(2\pi)^{-d} \int_0^t (t-r)^{-1/2} \int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} e^{i(x-z)y} \sum_{k=0}^{\infty} \frac{(-D(t-r)^{1/2}|y|^2)^k}{\Gamma(\frac{k+2}{2})} \bigg) \cdot$$

$$\cdot B(dr, dz), |y|^2 = y^2 = \sum_{j=1}^{d}. \tag{8.3}$$

Proof. The proof of formula (8.2) is given in Theorem 7.2. It remains to prove (8.3). For this purpose, let us first calculate

$$\mathcal{L}^{-1}\bigg(\frac{\sigma \mathcal{F}\mathcal{L}(\lambda, y)}{\tau\lambda + \lambda^{1/2} + |y|^2} \bigg). \tag{8.4}$$

Recall that the convolution $f * g$ of two functions $f, g : [0, \infty) \to \mathbb{R}$ is defined in the form

$$(f * g)(t) = \int_0^t f(t-r)g(r)dr, t \geq 0. \tag{8.4}$$

The convolution rules for the Laplace transform establish that

$$\mathcal{L}\bigg(\int_0^t f(t-r)g(r)dr \bigg) = \mathcal{L}f(\lambda)\mathcal{L}g(\lambda)$$

or

$$\int_0^t f(t-\omega)g(\omega)d\omega = \mathcal{L}^1(\mathcal{L}f(\lambda)\mathcal{L}g(\lambda))(t). \tag{8.5}$$

From (4.9) we obtain

$$\mathcal{L}^{-1}\bigg(\frac{1}{\tau\lambda + \lambda^{1/2} + d|y|^2} \bigg)(t) = \frac{1}{\sqrt{t}} E_{1/2,1/2}(-D(t+\sqrt{t})|y|^2) =$$

$$\sum_{k=0}^{\infty} \frac{(-D(t+\sqrt{t})|y|^2)}{\Gamma(\frac{k+1}{2})} = \Lambda(\lambda, y). \tag{8.6}$$

In other words

$$\frac{\sigma}{\tau\lambda+\lambda^{1/2}+d|y|^2}=\sigma\mathcal{L}(\Lambda(\lambda,y)).$$

Combining with (8.5) we find

$$\mathcal{L}^{-1}\bigg(\frac{\sigma}{\tau\lambda+\lambda^{1/2}+d|y|^2}\mathcal{F}(\mathcal{L}(\lambda,y))\bigg)(t)=$$

$$=\mathcal{L}^{-1}(\mathcal{L}(\sigma\Lambda(\lambda,y))\mathcal{L}(\mathcal{F}W(\lambda,y)))(t)=$$

$$=\sigma\int_0^t\Lambda(t-r,y\mathcal{F}(W(r,y)))dr. \tag{8.7}$$

Theorem proved.

Substituting now (8.7) into the formula

$$\mathcal{L}(u(t,y))=\mathcal{L}^{-1}\bigg(\frac{1}{\sqrt{\lambda}(\tau\lambda+\lambda^{1/2}+d|y|^2)}\bigg)(t,y)+$$

$$+\mathcal{L}^{-1}\bigg(\frac{\sigma\mathcal{L}(\mathcal{F}W(\lambda,y))}{\tau\lambda+\lambda^{1/2}+d|y|^2}\bigg)(t,y)$$

we get

$$\mathcal{F}(u(t,y))=E_{1/2,1}(-d(\sqrt{t}+t)|y|^2)+\sigma\int_0^t\Lambda(t-r,y)\mathcal{F}(r,y)dr.$$

Taking now the inverse Fourier transform we find

$$u(t,x)=\mathcal{F}^{-1}(E_{1/2,1}(-D(\sqrt{t}+t)|y|^2)(x)+$$

$$+\sigma F^{-1}\bigg(\int_0^t\Lambda(t-r,y))\mathcal{F}(W(r,y)dr)(x)\bigg)=$$

$$=\mathcal{F}^{-1}\bigg(\sum_{k=0}^{\infty}\frac{(-D(\sqrt{t}+t\tau)|y|^2)^k}{\Gamma(\frac{k+2}{2})}\bigg)+$$

$$+\sigma(2\pi)^{-d}\bigg|_0^t\int_{\mathbb{R}^D}\bigg(\int_{\mathbb{R}^d}e^{i(x-z)y}\Lambda(t-r,y)y\bigg)B(dr,dz)=$$

$$=(2\pi)^{-d}\int_{\mathbb{R}^2}e^{ixy}\sum_{k=0}^{\infty}\frac{(-D(\sqrt{t}+t\tau)|y|^2)^k}{\Gamma(\frac{k+2}{2})}dy+$$

$$\sigma(2\pi)^{-d}\int_0^t(t-\tau)^{1/2}\bigg(\int_{\mathbb{R}^d}(\int_{\mathbb{R}^2}e^{i(x-z)y}\sum_{k=0}^{\infty}\frac{(-D(t-r)^{1/2}|y|^2)^k}{\Gamma(\frac{k+2}{2})})B(dr,dz)\bigg).$$

In [19-23] the existence, singularity and stability of solutions of some classes of evolutionary stochastic differential equations are studied.

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