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Difference methods for infinite systems of parabolic functional differential equations

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Abstract. We consider the classical solutions of mixed problems for infinite, countable systems of parabolic partial functional differential equations. Difference methods of two types are constructed and convergence theorems are proved. In the first type, we approximate the exact solutions by solutions of infinite difference systems. Methods of second type consist in truncation of the infinite difference system, so that the resulting difference problem is finite and practically solvable. The proof of stability is based on a comparison technique with nonlinear estimates of the Perron type for the given functions. The comparison system is infinite. Parabolic problems with deviated variables and integro-differential problems can be obtained from the general model by specifying the given operators.

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1. Introduction

For any metric spaces X and Y, we write $C(X,Y)$ for the class of all continuous functions from X into Y, unless stated otherwise. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. The inequalities between vectors should be understood componentwise.

Denote by l^∞ the class of all real sequences $p = \{p_k\}_{k\in\mathbb{N}}$ having the property

$$\|p\|_\infty = \sup\{|p_k| : k \in \mathbb{N}\} < \infty.$$

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The class of all $n \times n$ matrices with real elements will be denoted by $M_{n\times n}$. For the norm in $\mathbb{R}^n$, let us choose

$$\|y\| = \sum_{j=1}^{m} |y_j|, \qquad y = (y_1, \ldots, y_n).$$

For r, $\bar{r} \in M_{n\times n}$, $[r_{ij}]_{i,j=1,\ldots,n}$, $[\bar{r}_{ij}]_{i,j=1,\ldots,n}$, we write $r \le \bar{r}$ if $\sum\limits_{i,j=1}^{n} (r_{ij} - \bar{r}_{ij})\lambda_i\lambda_j \le 0$ for any $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

Let $a > 0$, $d_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty)$, $d = (d_1, \ldots, d_n) \in \mathbb{R}^n_+$, and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$ be given where $b_j > 0$ for $1 \le j \le n$. Let us define

$$E = [0,a] \times (-b,b),\ D = [-d_0, 0] \times [-d,d],\ E_0 = [-d_0,0] \times [-b-d, b+d],$$
$$\partial_0 E = ([0,a] \times [-b-d, b+d]) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

Suppose that $z : E^* \to \mathbb{R}$ and $(t,x) \in \overline{E}$ are fixed, where $\overline{E}$ stands for the closure of E. We define the function $z_{(t,x)} : D \to \mathbb{R}$ by

$$z_{(t,x)}(\zeta,\xi) = z(t+\zeta, x+\xi), \quad (\zeta,\xi) \in D.$$

The function $z_{(t,x)}$ is the restriction of z to $[t-d_0,t] \times [x-d, x+d]$ and this restriction is shifted to D. For a domain $U \subset \mathbb{R}^{1+n}$ and a function $z : U \to l^\infty$ of the variables (t,x) we will write $\partial_t z = \{\partial_t z_k\}_{k\in\mathbb{N}}$, provided that respective derivatives exist.

Put $\Omega = E \times C(D, l^\infty) \times \mathbb{R}^n \times M_{n\times n}$ and let

$$f : \Omega \to l^\infty, \quad f = \{f^{(k)}\}_{k\in\mathbb{N}}, \qquad \varphi : E_0 \cup \partial_0 E \to l^\infty,$$
$$\alpha_0 = \{\alpha_{0,k}\}_{k\in\mathbb{N}} : \overline{E} \to l^\infty, \qquad \alpha' : \overline{E} \to l^\infty_n$$

be given. We introduce the notation $\alpha_k = (\alpha_{0,k}, \alpha'_k)$. It is required that $\alpha_{0,k}(t,x) \le t$ and $\alpha_k(t,x) \in \overline{E}$ for $(t,x) \in \overline{E}$, $k \in \mathbb{N}$. For a function $z : E^* \to l^\infty$, and for a point $(t,x) \in E$, we write

$$z_{\alpha(t,x)} = \{(z_k)_{\alpha_k(t,x)}\}_{k\in\mathbb{N}} \quad \text{and} \quad F[z](t,x) = F^{(k)}[z](t,x),$$

where

$$F^{(k)}[z](t,x) = f^{(k)}(t, x, z_{\alpha(t,x)}, \partial_x z_k(t,x), \partial_{xx} z_k(t,x)).$$

We deal with the following mixed problem:

$$\partial_t z(t,x) = F[z](t,x) \tag{1.1}$$
$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{1.2}$$

A function $v : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, is a classical solution of problem (1.1), (1.2) if

(i) $v \in C(E^*, l^\infty)$, derivatives $\partial_t v_k$, $\partial_{x_i} v_k$, $\partial_{x_i x_j} v_k$, $1 \le i,j \le n$, exist and are continuous on E for all $k \in \mathbb{N}$,

(ii) v satisfies (1.1) on E and the condition (1.2) holds true.

A classical solution $v = \{v_k\}_{k\in\mathbb{N}}$ of (1.1) is called a *parabolic solution* of (1.1) in E if for any two symmetric matrices r, $\bar{r} \in M_{n\times n}$, the inequality $r \le \bar{r}$ implies

$$f^{(k)}(t, x, v_{\alpha(t,x)}, \partial_x v_k(t, x), r) \le f^{(k)}(t, x, v_{\alpha(t,x)}, \partial_x v_k(t, x), \bar{r})$$

for $(t, x) \in E$, $k \in \mathbb{N}$.

Difference methods for nonlinear parabolic differential or functional differential problems were considered by many authors and under various assumptions. It is easy to construct an explicit Eulers type difference method which satisfies the consistency conditions on all sufficiently regular solutions of a differential functional problem. The main task in these investigations is to find a finite difference scheme which is stable. The method of difference inequalities and simple theorems on recurrent inequalities are used in the investigations of the stability of nonlinear difference or functional difference equations generated by parabolic problems, see, for example, [5, 7, 10]. Difference methods for (weakly or strongly coupled) parabolic systems with time delays were considered in [8, 9].

Within the last few years, numerous papers have been published, concerning various problems for infinite systems of parabolic partial functional differential equations. The monograph [1] contains up-to-date exposition of results concerning parabolic systems, including existence of solutions in Sobolev spaces. Various applications of infinite countable systems of parabolic partial integro-differential equations, such as the discrete coagulation-fragmentation model [12], are also listed in this monograph.

In this paper we use general ideas presented in [3, 6].

The paper is organized as follows. In section 2 we prove theorems on infinite systems of ordinary functional differential inequalities. Section 3 deals with a theoretical approximation method, based on an infinite system of difference equations. Numerical analysis of finite systems of functional difference equations, corresponding to the original problem, is presented in the section 4. Finally, numerical examples are given, covering both types of functional dependence admissible here, that is, an integral differential system and a system of equations with deviated variable.

2. Extremal solutions of functional differential systems

Let l^∞_+ be the set of sequences $p \in l^\infty$ such that $p_k \in \mathbb{R}_+$ for $k \in \mathbb{N}$. Put $I_0 = [-d_0, 0]$ and $I = (0, a)$. Denote by $C(I_0 \cup I, l^\infty_+)$ the space of all functions $w = \{w_k\}_{k\in\mathbb{N}}$ such that $w_k \in C(I_0 \cup I, \mathbb{R}_+)$. For such w we write $w'(t) = \{w'_k(t)\}_{k\in\mathbb{N}}$ and $D_-w(t) = \{D_-w_k(t)\}_{k\in\mathbb{N}}$ where $D_-w_k(t)$ stands for the left-hand lower Dini derivative of w_k at the point t.

For $\eta : I_0 \cup I \to \mathbb{R}$ and $t \in I$ we define $\eta_{(t)} : I_0 \to \mathbb{R}$ by $\eta_{(t)}(\tau) = \eta(t + \tau)$, $\tau \in I_0$. For $\eta : I_0 \cup I \to l^\infty$, $\eta = \{\eta_k\}_{k\in\mathbb{N}}$, the symbol $\eta_{(t)}$ should be understood componentwise.

By abuse of notation, let us write $C(I_0, l^\infty_+)$ for the class of all functions $\eta = \{\eta_k\}_{k\in\mathbb{N}}$ such that $\eta_k \in C(I_0, \mathbb{R}_+)$, $k \in \mathbb{N}$, and with finite norm

$$|\eta|_0 = \sup\{|\eta_k|_0 : k \in \mathbb{N}\} < +\infty,$$

where $|\eta_k|_0$ is the supremum norm of η_k in the space $C(I_0, \mathbb{R})$.

Put $\Xi = I \times C(I_0, l^\infty_+)$ and suppose that

$$\sigma = \{\sigma_k\}_{k\in\mathbb{N}}, \quad \sigma_k : \Xi \to \mathbb{R}_+ \quad \text{and} \quad \eta = \{\eta_k\}_{k\in\mathbb{N}}, \quad \eta_k : I_0 \to \mathbb{R}_+,$$

are given. We consider the Cauchy problem

$$w'(t) = \sigma(t, w_{(t)}), \tag{2.1}$$

$$w_{(0)} \equiv \eta. \tag{2.2}$$

Let p, $\tilde{p} \in l^\infty$, $p = \{p_k\}_{k\in\mathbb{N}}$, $\tilde{p} = \{\tilde{p}_k\}_{k\in\mathbb{N}}$. We will write

$$p \le \tilde{p} \qquad \text{if} \qquad p_k \le \tilde{p}_k \quad \text{for} \quad k \in \mathbb{N}. \tag{2.3}$$

Assumption H$_0$ $[\sigma]$. The function $\{\sigma_k\}_{k\in\mathbb{N}} = \sigma : \Xi \to l^\infty_+$, is such that $\sigma_k : \Xi \to \mathbb{R}_+$ is continuous for each $k \in \mathbb{N}$, and

1) the following monotonicity condition holds: if w, $\tilde{w} \in C(I_0, l^\infty_+)$ are such that $w(t) \le \tilde{w}(t)$ on I_0, then $\sigma(t, w) \le \sigma(t, \tilde{w})$,

2) there are a_0, $b_0 \in \mathbb{R}_+$ such that

$$\|\sigma(t, w)\|_\infty \le a_0 \|w\|_\infty + b_0 \quad \text{on} \quad \Xi.$$

Lemma 2.1. *Suppose that Assumption H_0 $[\sigma]$ is satisfied and $\eta = \{\eta_k\}_{k\in\mathbb{N}}$, $\eta \in C(I_0, l^\infty_+)$. Thus there exists the maximum solution*

$$\omega(\cdot, \eta) = \{\omega_k(\cdot, \eta)\}_{k\in\mathbb{N}}, \quad \omega_k(\cdot, \eta) : I_0 \cup I \to \mathbb{R}_+,$$

of the Cauchy problem (2.1), (2.2). Moreover, if $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$, $\varphi \in C(I_0 \cup I, l^\infty_+)$, and φ satisfies the system of functional differential inequalities

$$D_-\varphi \le \sigma(t, \varphi_{(t)}), \quad t \in I, \tag{2.4}$$

and the initial estimate holds

$$\varphi(t) \le \eta(t) \quad \textit{for} \quad t \in I_0, \tag{2.5}$$

then

$$\varphi(t) \le \omega(t, \eta) \quad \textit{for} \quad t \in I. \tag{2.6}$$

Proof. Take a function $\psi = \{\psi_k\}_{k\in\mathbb{N}}$, $\psi \in C(I_0 \cup I, l^\infty_+)$, and put

$$\Lambda_{k.\psi}(t, \zeta) = \sigma_k(t, P_k[t, \zeta, \psi]),$$

for $(t, \zeta) \in I \times C(I_0, \mathbb{R}_+)$ and

$$P_k[t, \zeta, \psi] = ((\psi_1)_{(t)}, \ldots, (\psi_{k-1})_{(t)}, \zeta, (\psi_{k+1})_{(t)}, \ldots).$$

There is (see [4]) the right-hand maximum solution $W_k[\psi]$ of the Cauchy problem

$$\xi'(t) = \Lambda_{k.\psi}(t, \xi_{(t)}), \quad \xi_{(0)} \equiv \eta_k,$$

and the solution is defined on $I_0 \cup I$. Put $W[\psi] = \{W_k[\psi]\}_{k\in\mathbb{N}}$. It follows from the monotonicity condition for σ and from a theorem on functional differential inequalities (see [4]) that for two functions ψ, $\tilde{\psi} \in C(I_0 \cup I, l^\infty_+)$ such that $\psi(t) \le \tilde{\psi}(t)$ on $I_0 \cup I$ we have

$$W[\psi](t) \le W[\tilde{\psi}](t) \quad \text{on} \quad I.$$

Denote by Δ the class of all functions $w = \{w_k\}_{k\in\mathbb{N}}$, $w \in C(I_0 \cup I, l^\infty_+)$, satisfying the differential inequality

$$D_- w(t) \le \sigma(t, w_{(t)}), \quad t \in I,$$

and the initial estimate $w(t) \le \eta(t)$ for $t \in I_0$. For every $k \in \mathbb{N}$ the family of functions $\{W_k[\psi]\}_{\psi\in\Delta}$ is bounded and equicontinuous on $I_0 \cup I$. Hence

$$\tilde{\omega}_k(t) = \sup\{W_k[\psi](t) : \psi \in \Delta\}, \quad t \in I_0 \cup I, k \in \mathbb{N},$$

exists and is a continuous function. Moreover, $\tilde{\omega}_k(t) = \eta_k(t)$ on I_0 for $k \in \mathbb{N}$. Put $\tilde{\omega} = \{\tilde{\omega}_k\}_{k\in\mathbb{N}}$. Thus $\tilde{\omega} \in C(I_0 \cup I, l^\infty_+)$ and $W[\psi](t) \le W[\tilde{\psi}](t)$ for $\psi \in \Delta$, $t \in I$. Therefore,

$$\tilde{\omega}(t) \le W[\tilde{\omega}](t) \quad \text{for} \quad t \in I. \tag{2.7}$$

On the other hand, we have

$$\frac{d}{dt} W_k[\tilde{\omega}](t) = \Lambda_{k.\tilde{\omega}}(t, (W_k[\tilde{\omega}])_{(t)}) = \sigma_k(t, P_k[t, W_k[\tilde{\omega}], \tilde{\omega}]).$$

Hence, by (2.7) and by the monotonicity condition for σ we obtain

$$\frac{d}{dt} W_k[\tilde{\omega}](t) \le \sigma_k(t, (W[\tilde{\omega}])_{(t)}), \quad t \in I, k \in \mathbb{N},$$

and consequently $W[\tilde{\omega}] \in \Delta$. This gives

$$W_k[\tilde{\omega}](t) \le \sup\{W_k[\psi](t) : \psi \in \Delta\} = \tilde{\omega}(t) \quad \text{for} \quad t \in I, k \in \mathbb{N}$$

and we conclude that

$$W[\tilde{\omega}](t) \le \tilde{\omega}(t) \quad \text{for} \quad t \in I. \tag{2.8}$$

Inequalities (2.7) and (2.8) imply $\tilde{\omega} = W[\tilde{\omega}]$. Thus $\tilde{\omega}$ is the right-hand maximum solution of (2.1), (2.2). It follows from (2.4), (2.5) that $\varphi \in \Delta$ and estimate (2.6) is proved. □

Let $V : C(D, l^\infty) \to C(I_0, l^\infty_+)$ be the operator defined by

$$\big(Vw\big)_k(t) = \max\{|w_k(t,x)| : x \in [-d, d]\}, \quad k \in \mathbb{N}$$

for $w \in C(D, l^\infty)$, $w = \{w_k\}_{k\in\mathbb{N}}$, and $t \in [-d_0, 0]$. For a function $w : A \to l^\infty$, $A \subset \mathbb{R}$, $w = \{w_k\}_{k\in\mathbb{N}}$, and for $a \in \mathbb{R}$ we will write $w \equiv a$ if $w_k \equiv a$ for $k \in \mathbb{N}$.

Assumption H $[f, \sigma]$**.** There is a comparison function

$$\sigma : I \times C(I_0, l^\infty_+) \to l^\infty_+, \qquad \sigma = \{\sigma_k\}_{k\in\mathbb{N}},$$

of variables (t, w), satisfying Assumption H_0 $[\sigma]$ and such that

1) $\sigma_k : I \times C(I_0, l^\infty_+) \to \mathbb{R}_+$ is non-decreasing with respect to t, $k \in \mathbb{N}$,
2) for $\theta \in C(I_0, l^\infty_+)$, $\theta \equiv 0$, holds $\sigma(\cdot, \theta) \equiv 0$,
3) the maximal solution of the Cauchy problem

$$\begin{aligned} \eta_k'(t) &= \sigma_k(t, \eta_{(t)}), \quad t \in I,\ k \in \mathbb{N}, \\ \eta_{(0)}(t) &\equiv 0 \end{aligned}$$

is $\{\tilde\eta_k\}_{k\in\mathbb{N}} = \tilde\eta : I_0 \cup I \to l^\infty$, $\tilde\eta \equiv 0$,

4) for $\{\varepsilon_k\}_{k\in\mathbb{N}} = \varepsilon \in l^\infty_+$, the maximum solution $\omega(\cdot, \varepsilon) : I_0 \cup I \to l^\infty_+$ of the Cauchy problem

$$\begin{aligned} \eta_k'(t) &= \sigma_k(t, \eta_{(t)} + \varepsilon) + \varepsilon_k, && t \in I,\ k \in \mathbb{N}, \\ \eta(t) &= \varepsilon, && t \in I_0, \end{aligned}$$

is such that

$$\lim_{k\to\infty} \varepsilon_k = 0 \qquad \text{implies} \qquad \lim_{k\to\infty} \omega_k(\cdot, \varepsilon) = 0 \quad \text{uniformly on} \quad I,$$

5) for the above Cauchy problem, $\lim_{\varepsilon\to 0} \|\omega(t, \varepsilon)\|_\infty = 0$ uniformly on I,
6) the upper bounds on growth of f with respect to the functional argument are

$$|f^{(k)}(t, x, w, q, r) - f^{(k)}(t, x, \bar w, q, r)| \le \sigma_k(t, V(w - \bar w)), \quad k \in \mathbb{N},$$

for (t, x, w, q, r), $(t, x, \bar w, q, r) \in \Omega$.

3. Infinite systems of difference equations

We formulate a difference problem, corresponding to (1.1), (1.2). We define a uniform mesh on E^* in the following way. Let $h = (h_0, h')$, $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. Then we choose nodal points $(t^{(r)}, x^{(m)})$ by

$$t^{(r)} = rh_0, \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1h_1, \ldots, m_nh_n),$$

for $(r, m) \in \mathbb{Z}^{1+n}$. We write H for the set of all h such that there is $-N_0 \in \mathbb{Z}$ with the property $-N_0h_0 = -d_0$ and there is $N_j \in \mathbb{N}$ satisfying $N_jh_j = b_j$ for each index j such that $d_j = 0$. Let $t^{(K_0)}$ be the last temporal node, that is, $t^{(K_0)} \le a < t^{(K_0+1)}$. We put $\|h\| = h_0 + h_1 + \ldots + h_n$. Using the notion of the set of all nodal points in

$\mathbb{R}^{1+n}$, $\mathbb{R}^{1+n}_h = \{(t^{(r)}, x^{(m)}) : (r,m) \in \mathbb{Z}^{1+n}\}$, we define various parts of the discrete domain simply by:

$$E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_h, \quad E_h = E \cap \mathbb{R}^{1+n}_h, \quad \partial_0 E_h = \partial_0 E \cap \mathbb{R}^{1+n}_h,$$
$$E^*_h = E_{0.h} \cup E_h \cup \partial_0 E_h.$$

The discrete counterparts of sets I_0, I and $I_0 \cup I$:

$$I_{0.h} = \Big\{t^{(r)} : -N_0 \le r \le 0\Big\}, \quad I_h = \Big\{t^{(r)} : 0 \le r \le K_0\Big\}, \quad I^*_h = I_{0.h} \cup I_h$$

will also be needed. The difference equation will be considered at the points lying in

$$E'_h = \Big\{(t^{(r)}, x^{(m)}) \in E_h : 0 \le r \le K_0 - 1\Big\}.$$

We now describe the interpolation operator T_h, that has been presented in [2]. T_h maps real functions defined on the mesh E^*_h into real functions defined on E^*. Suppose that $z : E^*_h \to \mathbb{R}$. For $(t,x) \in E^*$, three cases will be distinguished.

(i) There is $(r,m) \in \mathbb{Z}^{1+n}$ such that $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E^*_h$ and $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$, where $m+1 = (m_1+1, \dots, m_n+1)$. We define

$$\begin{aligned}(T_h z)(t,x) = &\Big(1 - \frac{t - t^{(r)}}{h_0}\Big) \sum_{\mu \in \{0,1\}^n} z^{(r, m+\mu)} \Big(\frac{x - x^{(m)}}{h}\Big)^{\mu} \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-\mu} \\ &+ \Big(\frac{t - t^{(r)}}{h_0}\Big) \sum_{\mu \in \{0,1\}^n} z^{(r+1, m+\mu)} \Big(\frac{x - x^{(m)}}{h}\Big)^{\mu} \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-\mu},\end{aligned}$$

using the notation

$$\begin{aligned}\Big(\frac{x - x^{(m)}}{h}\Big)^{\mu} &= \prod_{j=1}^{n} \Big(\frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{\mu_j}, \\ \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-\mu} &= \prod_{j=1}^{n} \Big(1 - \frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{1-\mu_j}.\end{aligned}$$

We adopt here the convention that $0^0 = 1$.

(ii) Suppose that x lies near the boundary of $[-b-d, b+d]$, namely that $N_j h_j \le |x_j| \le b_j + d_j < (N_j+1) h_j$ for some j, $1 \le j \le n$. Then we put $T_h z(t,x) = T_h z(t, \tilde{x})$, where

$$\tilde{x}_j = \begin{cases} x_j, & |x_j| \le N_j h_j \\ -N_j h_j, & x_j < -N_j h_j \\ N_j h_j, & x_j > N_j h_j, \end{cases} \qquad 1 \le j \le n.$$

(iii) The last case is $K_0h_0 < t \le a$. Then we set $T_h z(t,x) = T_h z(K_0h_0, x)$.

Note that T_h is a linear operator, mapping real functions defined on E_h^* into continuous functions defined on E^*. Furthermore, since $T_h z_h$ interpolates z_h by use of convex combinations of z_h's values, the equality

$$\max_{t\le t^{(r)},\, x\in[-b-d,b+d]} |(T_h z_h)(t,x)| = \max_{i\le r,\, -K\le m\le K} \left| z_h^{(i,m)} \right| \tag{3.1}$$

holds for $-N_0 \le r \le K_0$.

The following lemma is a direct consequence of the definition of T_h.

Lemma 3.1. *Suppose that $v : E^* \to \mathbb{R}$ is a continuous function, and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ its modulus of continuity. Then*

$$|(T_h v_h - v)(t,x)| \le \omega(\|h\|),$$

where $v_h = v\big|_{E_h^*}$.

The symbol $T_h z$ for $z : E_h^* \to l^\infty$ should be understood componentwise.

Write $e_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{Z}^n$ with the 1 standing on the i-th place. Put $J = \{(i,j) : 1 \le i,j \le n, i \ne j\}$, and suppose that for each $k \in \mathbb{N}$ we have defined two disjoint (one possibly empty) subsets $J_{k.+}$, $J_{k.-}$ of J, such that $J_{k.+} \cup J_{k.-} = J$. We assume that $(i,j) \in J_{k.+}$ if $(j,i) \in J_{k.+}$. Let $z : E^* \to l^\infty$ and $(t^{(r)}, x^{(m)}) \in E_h'$. The definitions of difference operators, involved in our difference scheme, will be given with the aid of

$$\delta_i^+ z_k^{(r,m)} = \frac{1}{h_i}[z_k^{(r,m+e_i)} - z_k^{(r,m)}] \quad \text{and} \quad \delta_i^- z_k^{(r,m)} = \frac{1}{h_i}[z_k^{(r,m)} - z_k^{(r,m-e_i)}],$$

for $1 \le i \le n$, $k \in \mathbb{N}$. Then, let δ_0, $\delta = (\delta_1,\dots,\delta_n)$, $[\delta^{(2)}{}_{ij}]_{i,j=1,\dots,n}$ be defined by

$$\begin{aligned}
\delta_0 z_k^{(r,m)} &= \frac{1}{h_0}[z_k^{(r+1,m)} - z_k^{(r,m)}], \\
\delta_i z_k^{(r,m)} &= \frac{1}{2}[\delta_i^+ + \delta_i^-] z_k^{(r,m)}, \;\; 1 \le i \le n,
\end{aligned}$$

$$\begin{aligned}
\delta_{ii}^{(2)} z_k^{(r,m)} &= \delta_i^+ \delta_i^- z_k^{(r,m)}, \;\; 1 \le i \le n, \\
\delta_{ij}^{(2)} z_k^{(r,m)} &= \frac{1}{2}[\delta_i^+ \delta_j^- + \delta_i^- \delta_j^+] z_k^{(r,m)} \quad \text{for} \quad (i,j) \in J_{k.-}, \\
\delta_{ij}^{(2)} z_k^{(r,m)} &= \frac{1}{2}[\delta_i^+ \delta_j^+ + \delta_i^- \delta_j^-] z_k^{(r,m)} \quad \text{for} \quad (i,j) \in J_{k.+}.
\end{aligned}$$

Let us write $F_h^{(k)}[z]^{(r,m)}$ as a short for

$$f^{(k)}(t^{(r)}, x^{(m)}, (T_h z)_{\alpha(t^{(r)},x^{(m)})}, \delta z_k^{(r,m)}, \delta^{(2)} z_k^{(r,m)}).$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to l^\infty$, we consider the difference-functional problem

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)}, \tag{3.2}$$

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h, \tag{3.3}$$

where $F_h[z]^{(r,m)} = \{F_h^{(k)}[z]^{(r,m)}\}_{k\in\mathbb{N}}$ and $\delta_0 z^{(r,m)} = \{\delta_0 z_k^{(r,m)}\}_{k\in\mathbb{N}}$. It is evident that there exists exactly one solution $u_h = \{u_{h.k}\}_{k\in\mathbb{N}}$ of (3.2)-(3.3).

The comparison technique used in the proof of convergence (Theorem 3.2) requires analogues of already defined operators: T_h and the Hale operator, for functions defined on I_h^*. First, we introduce the interpolation operator $T_{h.0}$, mapping real functions defined on I_h^* into real functions defined on $I_0 \cup I$. For $\omega : I_h^* \to \mathbb{R}$ and $t \in I_0 \cup I$, we put

$$(T_{h.0}\,\omega)(t) = \frac{t - t^{(r)}}{h_0}\omega^{(r+1)} + \left(1 - \frac{t - t^{(r)}}{h_0}\right)\omega^{(r)},$$

when $t^{(r)} \le t \le t^{(r+1)}$ for some $-N_0 \le r \le K_0 - 1$, and

$$(T_{h.0}\,\omega)(t) = \omega^{(K_0)},$$

when $t^{(K_0)} < t < a$. We define $T_{h.0}\,\omega$ for $\omega : I_{0.h} \to \mathbb{R}$ in a similar way. The symbol $T_{h.0}\,\omega$, for $\omega : I_h^* \to l^\infty$ or $\omega : I_{0.h} \to l^\infty$ should be understood componentwise.

Next, for $0 \le r \le K_0$, we define the discrete Hale operator, mapping real functions ω defined on the mesh I_h^* into real functions $\omega_{[r]}$ defined on $I_{0.h}$, by: $\omega_{[r]}(\tau) = \omega(t^{(r)} + \tau)$, $\tau \in I_{0.h}$. When $\omega : I_h^* \to l^\infty$, the symbol $\omega_{[r]}$ should be understood componentwise.

It is clear that for $\omega : I_h^* \to \mathbb{R}$ and $0 \le r \le K_0$

$$(T_{h.0}\,\omega)_{(t^{(r)})} = T_{h.0}\,\omega_{[r]}. \tag{3.4}$$

Lemma 3.2. *If $z : E_h^* \to \mathbb{R}$ and $\omega : I_h^* \to \mathbb{R}_+$ is defined by*

$$\omega^{(r)} = \max\left\{|z^{(i,m)}| : i \le r,\ x^{(m)} \in [-b-d, b+d]\right\}, \quad t^{(r)} \in I_h^*,$$

then

$$V(T_h z)_{(t,x)} \le (T_{h.0}\,\omega)_{(t)} \quad \text{for} \quad (t,x) \in E. \tag{3.5}$$

Proof. Put $A = [-b-d, b+d]$ and $A_r = [-d_0, t^{(r)}] \times A$. With this notation, (3.1) reads

$$\max_{(t,x)\in A_r} |(T_h z)(t,x)| \le \omega^{(r)}. \tag{3.6}$$

Moreover, straight from the definition of T_h,

$$(T_h z)(t,x) = \frac{t - t^{(r)}}{h_0}(T_h z)(t^{(r+1)}, x) + \left(1 - \frac{t - t^{(r)}}{h_0}\right)(T_h z)(t^{(r)}, x).$$

Hence, and by (3.6),

$$
\begin{aligned}
&\max_{x\in A}|(T_h z)(t,x)| \\
&\quad \le \frac{t-t^{(r)}}{h_0}\max_{x\in A}|(T_h z)(t^{(r+1)},x)| + \left(1-\frac{t-t^{(r)}}{h_0}\right)\max_{x\in A}|(T_h z)(t^{(r)},x)| \\
&\quad \le \frac{t-t^{(r)}}{h_0}\max_{(t,x)\in A_{r+1}}|(T_h z)(t,x)| + \left(1-\frac{t-t^{(r)}}{h_0}\right)\max_{(t,x)\in A_r}|(T_h z)(t,x)| \\
&\quad \le \frac{t-t^{(r)}}{h_0}\omega^{(r+1)} + \left(1-\frac{t-t^{(r)}}{h_0}\right)\omega^{(r)} \\
&\quad = (T_{h.0}\,\omega)(t).
\end{aligned}
$$

By the definition of V and by the above inequality,

$$
\begin{aligned}
(V(T_h z)_{(t,x)})(s) &= \max_{y\in[-d,d]}|(T_h z)(t+s,x+y)| \\
&\le \max_{y\in A}|(T_h z)(t+s,y)| \le (T_{h.0}\,\omega)(t+s) = (T_{h.0}\,\omega)_{(t)}(s),
\end{aligned}
$$

for $s \in I_0$, which proves (3.5). □

Corollary 3.1. *If $z : E_h^* \to l^\infty$ and $\omega : I_h^* \to l_+^\infty$ is defined by*

$$
\omega_k^{(r)} = \max\left\{|z_k^{(i,m)}| : i \le r,\ x^{(m)} \in [-b-d, b+d]\right\}, \quad t^{(r)} \in I_h^*,
$$

then

$$
V(T_h z)_{\alpha^{(r,m)}} \le T_{h.0}\,\omega_{[r]}, \quad (t^{(r)}, x^{(m)}) \in E_h. \tag{3.7}
$$

Proof. Clearly, monotonicity of ω implies

$$
(T_{h.0}\,\omega)_{(t_1)} \le (T_{h.0}\,\omega)_{(t_2)} \quad \text{for} \quad t_1 \le t_2. \tag{3.8}
$$

The assertion follows from the condition $\alpha_{0,k}(t,x) \le t$, $k \in \mathbb{N}$, and from relations (3.4), (3.8). □

Theorem 3.1. *Suppose that Assumption $H\,[f,\sigma]$ is fulfilled, and $\omega : I_h^* \to l_+^\infty$ satisfies the recurrent inequality*

$$
\omega_k^{(r+1)} \le \omega_k^{(r)} + h_0\sigma_k(t^{(r)}, T_{h.0}\,\omega_{[r]}) + h_0\gamma(h), \tag{3.9}
$$

for $0 \le r \le K_0 - 1$,

$$
\|\omega^{(r)}\|_\infty \le \gamma(h) \qquad \text{for} \quad t^{(r)} \in I_{0.h}, \tag{3.10}
$$

where $\gamma : H \to \mathbb{R}_+$, $\lim_{h\to 0}\gamma(h) = 0$, and the steps of the mesh are small enough to fulfil

$$
h_0\, \exp[a_0 a](a_0 + b_0 + (1+2a_0)\gamma(h)) \le 1, \tag{3.11}
$$

with a_0, b_0 from Assumption H_0 $[\sigma]$. Then there is $\tilde{\gamma} : H \to \mathbb{R}_+$ such that

$$\|\omega^{(r)}\|_\infty \le \tilde{\gamma}(h) \quad \textit{for} \quad t^{(r)} \in I_h \qquad\qquad \textit{and} \quad \lim_{h\to 0} \tilde{\gamma}(h) = 0. \tag{3.12}$$

Proof. Consider the Cauchy problem

$$\eta_k'(t) = \sigma_k(t, \eta_{(t)} + C(h)) + \gamma(h), \quad t \in I, \quad k \in \mathbb{N}, \tag{3.13}$$

$$\eta_{(0)} \equiv \gamma(h) \quad \text{on} \quad I_0, \tag{3.14}$$

where

$$C(h) = h_0 \exp[a_0 a](a_0 + b_0 + (1 + 2a_0)\gamma(h)), \tag{3.15}$$

and a_0, b_0 are such that $\|\sigma(t, w)\|_\infty \le a_0\|w\|_\infty + b_0$ on Ξ (see Assumption H_0 $[\sigma]$). Since the function σ with additions $\gamma(h)$ and $C(h)$ still satisifies Assumption H_0 $[\sigma]$, the Lemma 2.1 assures the existence of maximum solution $\eta_h = \{\eta_{h.k}\}_{k\in\mathbb{N}}$ of (3.13), (3.14) and η_h is defined on $I_0 \cup I$. The condition (3.11) implies $(C(h))_k \le 1$, $k \in \mathbb{N}$, and, consequently,

$$\eta_{h.k}'(t) \le \sigma_k(t, (\eta_h)_{(t)} + 1) + \gamma(h), \quad k \in \mathbb{N}.$$

Since $\eta_{h.k} : I_0 \cup I \to \mathbb{R}$, $k \in \mathbb{N}$, are differentiable, convex, and such that $\|\eta_h'(t)\|_\infty$ is finite, we have $D_-\|\eta_h(t)\|_\infty \le \|\eta_h'(t)\|_\infty$ on I. In view of this and of the Assumption H_0 $[\sigma]$, the function $\psi(t) = \|\eta_h(t)\|_\infty$ satisfies

$$D_-\psi(t) \le a_0\psi(t) + a_0 + b_0 + (1 + a_0)\gamma(h).$$

Hence

$$\psi(t) \le a_0^{-1}(\exp[a_0 t] - 1)(a_0 + b_0 + (1 + a_0)\gamma(h)) + \gamma(h)\exp[a_0 t]$$

and, consequently,

$$\eta_{h.k}'(t) \le \exp[a_0 t](a_0 + b_0 + (1 + 2a_0)\gamma(h)), \quad k \in \mathbb{N},$$

on $I_0 \cup I$. Majorizing the right-hand side by taking $t = a$, we obtain (recall (3.15)):

$$h_0\eta_{h.k}' \le C(h) \quad \text{on} \quad I_0 \cup I, \quad k \in \mathbb{N},$$

and hence

$$\eta_{h.k} + C(h) \ge T_{h.0}\,\widehat{\eta}_{h.k} \quad \text{on} \quad I_0 \cup I, \quad k \in \mathbb{N},$$

where $\widehat{\eta}_h$ is the restriction of η_h to I_h^*. The function η_h satisfies the difference inequality

$$\begin{aligned} \eta_{h.k}^{(r+1)} &\ge \eta_{h.k}^{(r)} + h_0\sigma_k(t^{(r)}, (\eta_h)_{(t^{(r)})} + C(h)) + h_0\gamma(h) \\ &\ge \eta_{h.k}^{(r)} + h_0\sigma_k(t^{(r)}, T_{h.0}\,(\widehat{\eta}_h)_{[r]}) + h_0\gamma(h) \end{aligned} \tag{3.16}$$

for $0 \le r \le K_0 - 1$ and $k \in \mathbb{N}$. By the (3.16), (3.14) and (3.9), (3.10) we have

$$\omega^{(r)} \le \eta_h^{(r)} \quad \text{for} \quad 0 \le r \le K_0. \tag{3.17}$$

Moreover, by Lemma 2.1 and by the condition 5) of Assumption H $[f, \sigma]$,

$$\lim_{h\to 0} \|\eta_h(t)\|_\infty = 0, \quad \text{uniformly on} \quad I.$$

Thus we get (3.12) for $\tilde{\gamma}(h) = \lim_{t\to a^-} \|\eta_h(t)\|_\infty$. This completes the proof. □

Assumption H $[f, H]$**.** The function $f : \Omega \to l^\infty$ of the variables (t, x, w, q, r), where $q = (q_1, \ldots, q_n)$ and $[r_{ij}]_{i,j=1,\ldots,n}$, satisfies the conditions:

1) for $k \in \mathbb{N}$, $f^{(k)} \in C(\Omega, \mathbb{R})$ and there is $B \in \mathbb{R}_+$ such that

$$\|f(P)\|_\infty \le B \quad \text{for} \quad P \in \Omega,$$

2) the derivatives

$$\partial_q f^{(k)} = (\partial_{q_1} f^{(k)}, \ldots, \partial_{q_n} f^{(k)}), \quad \partial_r f^{(k)} = [\partial_{r_{ij}} f^{(k)}]_{i,j=1,\ldots,n}$$

exist on Ω and are continuous with respect to (q, r) for each fixed $(t, x, w) \in E \times C(D, l^\infty)$,

3) the matrix $\partial_r f^{(k)}$ is symmetric and

$$\partial_{r_{ij}} f^{(k)}(P) \ge 0 \quad \text{for} \quad (i,j) \in J_{k.+}, \quad \partial_{r_{ij}} f^{(k)}(P) \le 0 \quad \text{for} \quad (i,j) \in J_{k.-}, \tag{3.18}$$

$$1 - 2h_0 \sum_{i=1}^n \frac{1}{h_i^2} \partial_{r_{ii}} f^{(k)}(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |\partial_{r_{ij}} f^{(k)}(P)| \ge 0, \tag{3.19}$$

$$-\frac{1}{2} |\partial_{q_i} f^{(k)}(P)| + \frac{1}{h_i} \partial_{r_{ii}} f^{(k)}(P) - \sum_{\substack{j=1 \\ j\ne i}}^n \frac{1}{h_j} |\partial_{r_{ij}} f^{(k)}(P)| \ge 0, \ \ 1 \le i \le n, \tag{3.20}$$

for $P \in \Omega$.

Remark 3.1. *Note that continuity and monotonicity of σ, together with the condition $\sigma_k(t, \theta) = 0$, $k \in \mathbb{N}$, imply continuity of $\sigma(t, \cdot)$ at point the (t, θ). This continuity is uniform with respect to $t \in I$.*

Theorem 3.2. *Suppose that Assumptions H $[f, H]$, H $[f, \sigma]$ are satisfied and*

1) there is $A \in \mathbb{R}_+$ such that $\|\varphi_h^{(r,m)}\|_\infty \le A$ on $E_{0.h} \cup \partial_0 E_h$,

2) $u_h : E_h^* \to l^\infty$, $u_h = \{u_{h.k}\}_{k\in\mathbb{N}}$, is a solution of (3.2) with initial boundary condition (3.3) given by $\{\varphi_{h.k}\}_{k\in\mathbb{N}} = \varphi_h$, *3) there is* $\beta^{(0)} : H \to \mathbb{R}_+$ *such that*

$$\|\varphi_h^{(r,m)} - \varphi(t^{(r)}, x^{(m)})\|_\infty \le \beta^{(0)}(h) \quad on \quad E_{0.h} \cup \partial_0 E_h \tag{3.21}$$

and $\lim_{h\to 0} \beta^{(0)}(h) = 0$,

4) $\tilde{v} : E^* \to l^\infty$ *is a solution of* (1.1), (1.2),

5) the functions $\tilde{v}_k$, $\partial_t \tilde{v}_k$, $\partial_{xx} \tilde{v}_k$, $k \in \mathbb{N}$, *are equicontinuous,*

6) there exists $C_0 \in \mathbb{R}_+$ *such that* $h_i h_j^{-1} \le C_0$ *for* $1 \le i, j \le n$.

Then, there exists $\tilde{\gamma} : H \to \mathbb{R}_+$ *such that*

$$\|u_h^{(r,m)} - \tilde{v}(t^{(r)}, x^{(m)})\|_\infty \le \tilde{\gamma}(h) \quad on \quad E_h^* \tag{3.22}$$

and $\lim_{h\to 0} \tilde{\gamma}(h) = 0$.

Proof. It is obvious that the condition 1), together with 1) of Assumption H $[f, H]$, imply the existence of u_h with values in l^∞. Precisely, the estimate $\|u_h^{(r,m)}\|_\infty \le A + t^{(r)} B$ holds on E_h.

Put $\tilde{v}_h = \{\tilde{v}_{h.k}\}_{k\in\mathbb{N}}$ where $\tilde{v}_{h.k}$ is the restriction of $\tilde{v}_k$ to E_h^*, $k \in \mathbb{N}$. We have on E'_h

$$\delta_0 \tilde{v}_h^{(r,m)} = F_h[\tilde{v}_h]^{(r,m)} + \Gamma_h^{(r,m)} \tag{3.23}$$

where $\Gamma_h^{(r,m)} = \{\Gamma_{h.k}^{(r,m)}\}_{k\in\mathbb{N}}$. Hence, by Remark 3.1 and by the regularity of $\tilde{v}$ and Lemma 3.1, there is $\beta^{(1)} : H \to \mathbb{R}_+$ such that

$$\|\Gamma_{h.k}^{(r,m)}\|_\infty \le \beta^{(1)}(h) \quad \text{on} \quad E'_h$$

and $\lim_{h\to 0} \beta^{(1)}(h) = 0$. Let the function $\varepsilon_h : E_h^* \to l^\infty$, $\varepsilon_h = \{\varepsilon_{h.k}\}_{k\in\mathbb{N}}$, be given by $\varepsilon_h = u_h - \tilde{v}_h$. Define $\omega_h : I_h \to l_+^\infty$, $\omega_h = \{\omega_{h.k}\}_{k\in\mathbb{N}}$, by

$$\omega_{h.k}^{(r)} = \max\left\{\beta^{(0)}(h), \quad \max\left\{|\varepsilon_{h.k}^{(i,m)}| : i \le r, x^{(m)} \in [-b-d, b+d]\right\}\right\},$$

for $0 \le r \le K_0$, $k \in \mathbb{N}$.

Our next goal is to estimate the function $\Psi = V\left[(T_h u_h - T_h \tilde{v}_h)_{\alpha^{(r,m)}}\right]$. We conclude from Corollary 3.1 that

$$\Psi \le V[(T_h \varepsilon_h)_{\alpha^{(r,m)}}] \le T_{h.0}\, \omega_{h[r]}. \tag{3.24}$$

Subtracting (3.2) and (3.23) by sides,

$$\begin{aligned}
\varepsilon_{h.k}^{(r+1,m)} &= \varepsilon_{h.k}^{(r,m)} + h_0 F_h^{(k)}[u_h]^{(r,m)} - h_0 F_h^{(k)}[\tilde{v}_h]^{(r,m)} - h_0 \Gamma_{h.k}^{(r,m)} \\
&= \varepsilon_{h.k}^{(r,m)} + h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h u_h)_{\alpha(t^{(r)},x^{(m)})}, \delta u_{h.k}^{(r,m)}, \delta^{(2)} u_{h.k}^{(r,m)}) \\
&\qquad - h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h \tilde{v}_h)_{\alpha(t^{(r)},x^{(m)})}, \delta u_{h.k}^{(r,m)}, \delta^{(2)} u_{h.k}^{(r,m)}) \\
&\qquad + h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h \tilde{v}_h)_{\alpha(t^{(r)},x^{(m)})}, \delta u_{h.k}^{(r,m)}, \delta^{(2)} u_{h.k}^{(r,m)}) \\
&\quad - h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h \tilde{v}_h)_{\alpha(t^{(r)},x^{(m)})}, \delta \tilde{v}_{h.k}^{(r,m)}, \delta^{(2)} \tilde{v}_{h.k}^{(r,m)}) - h_0 \Gamma_{h.k}^{(r,m)} \\
&= \varepsilon_{h.k}^{(r,m)} + h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h u_h)_{\alpha(t^{(r)},x^{(m)})}, \delta u_{h.k}^{(r,m)}, \delta^{(2)} u_{h.k}^{(r,m)}) \\
&\qquad - h_0 f^{(k)}(t^{(r)}, x^{(m)}, (T_h \tilde{v}_h)_{\alpha(t^{(r)},x^{(m)})}, \delta u_{h.k}^{(r,m)}, \delta^{(2)} u_{h.k}^{(r,m)}) \\
&\quad + h_0 \sum_{i=1}^{n} \partial_{q_i} f^{(k)}(Q) \delta_i \varepsilon_{h.k}^{(r,m)} + h_0 \sum_{i,j=1}^{n} \partial_{r_{ij}} f^{(k)}(Q) \delta_{ij}^{(2)} \varepsilon_{h.k}^{(r,m)} - h_0 \Gamma_{h.k}^{(r,m)},
\end{aligned}$$

where $Q \in \Omega$ is an intermediate point. We have, in view of condition 3) of Assumption H $[f, H]$, and definitions of difference operators (see page 60),

$$\begin{aligned}
&\Big| \varepsilon_{h.k}^{(r,m)} + h_0 \sum_{i=1}^{n} \partial_{q_i} f^{(k)}(Q) \delta_i \varepsilon_{h.k}^{(r,m)} + h_0 \sum_{i,j=1}^{n} \partial_{r_{ij}} f^{(k)}(Q) \delta_{ij}^{(2)} \varepsilon_{h.k}^{(r,m)} \Big| \\
&\le |\varepsilon_{h.k}^{(r,m)}| \Big(1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \partial_{r_{ii}} f^{(k)}(Q) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |\partial_{r_{ij}} f^{(k)}(Q)| \Big) \\
&+ h_0 \sum_{i=1}^{n} |\varepsilon_{h.k}^{(r,m+e_i)}| \Big(\frac{1}{2h_i} \partial_{q_i} f^{(k)}(Q) + \frac{1}{h_i^2} \partial_{r_{ii}} f^{(k)}(Q) - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{h_i h_j} |\partial_{r_{ij}} f^{(k)}(Q)| \Big) \\
&+ h_0 \sum_{i=1}^{n} |\varepsilon_{h.k}^{(r,m-e_i)}| \Big(-\frac{1}{2h_i} \partial_{q_i} f^{(k)}(Q) + \frac{1}{h_i^2} \partial_{r_{ii}} f^{(k)}(Q) - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{h_i h_j} |\partial_{r_{ij}} f^{(k)}(Q)| \Big) \\
&+ h_0 \sum_{(i,j) \in J_{k.-}} |\partial_{r_{ij}} f^{(k)}(Q)| \frac{1}{2h_i h_j} |\varepsilon_{h.k}^{(r,m-e_i+e_j)} + \varepsilon_{h.k}^{(r,m+e_i-e_j)}| \\
&+ h_0 \sum_{(i,j) \in J_{k.+}} |\partial_{r_{ij}} f^{(k)}(Q)| \frac{1}{2h_i h_j} |\varepsilon_{h.k}^{(r,m-e_i-e_j)} + \varepsilon_{h.k}^{(r,m+e_i+e_j)}| \\
&\le \omega_{h.k}^{(r)}.
\end{aligned}$$

Now, from Assumption H $[f, H]$ and from the monotonicity of σ with respect to the functional variable, follows

$$|\varepsilon_{h.k}^{(r+1,m)}| \le \omega_{h.k}^{(r)} + h_0 \sigma_k(t^{(r)}, T_{h.0}\, \omega_{h[r]}) + h_0 \beta^{(1)}(h)$$

for each $-K \le m \le K$ and $k \in \mathbb{N}$. Put $\gamma(h) = \max\{\beta^{(0)}(h), \beta^{(1)}(h)\}$. We have

$\lim_{h\to 0} \gamma(h) = 0$ and

$$\omega_{h.k}^{(r+1)} \le \omega_{h.k}^{(r)} + h_0\sigma_k(t^{(r)}, T_{h.0}\,\omega_{h[r]}) + h_0\gamma(h) \tag{3.25}$$

for $0 \le r \le K_0 - 1$, $k \in \mathbb{N}$, and

$$\|\omega_h^{(r)}\|_\infty \le \gamma(h) \quad \text{on} \quad I_{0.h}. \tag{3.26}$$

Hence, assertion follows from the Theorem 3.1. □

4. Finite systems of difference equations

Let us define

$$E^*[\,t\,] = [-d_0, t] \times [-b-d, b+d], \qquad t \in I_0 \cup I$$

and

$$\|z\|_{[\,t\,]} = \sup\{|z(\tau,x)| : (\tau,x) \in E^*[\,t\,]\}$$

for $z : E^* \to \mathbb{R}$.

Lemma 4.1. *Suppose that $\tilde\sigma : \Xi \to l^\infty_+$ fulfils Assumption $H_0\,[\sigma]$ and is non-decreasing with respect to t, and the function $f : \Omega \to l^\infty$, $f = \{f^{(k)}\}_{k\in\mathbb{N}}$, satisfies the estimates*

$$|f^{(k)}(t,x,w,0,0)| \le \tilde\sigma_k(t, Vw), \quad k \in \mathbb{N},$$

for $(t,x,w) \in E \times C(D, l^\infty)$. Suppose that $u = \{u_k\}_{k\in\mathbb{N}}$ is a classical parabolic solution of (1.1) in E satisfying the homogeneous initial boundary condition

$$u_k(t,x) = 0 \quad \text{on} \quad E_0 \cup \partial_0 E, \quad k \in \mathbb{N}$$

and $\{\omega_k\}_{k\in\mathbb{N}} = \omega : I_0 \cup I \to l^\infty_+$ is the maximum solution of the Cauchy problem

$$\eta'(t) = \tilde\sigma(t, \eta_{(t)}), \qquad \eta_{(0)} \equiv 0.$$

Then, for $k \in \mathbb{N}$,

$$\|u_k\|_{[\,t\,]} \le \omega_k(t), \quad t \in I.$$

Proof. Put $W_k(t) = \|u_k\|_{[\,t\,]}$, $k \in \mathbb{N}$, $t \in I_0 \cup I$, and let $W = \{W_k\}_{k\in\mathbb{N}}$. We will prove that for fixed $k \in \mathbb{N}$,

$$D_- W_k(t) \le \tilde\sigma_k(t, W_{(t)})$$

for $t \in I$. Let us fix $\tilde t \in I$. Since $W_k \equiv 0$ on I_0, W_k is non-decreasing, and $\tilde\sigma_k$ attains non-negative values, we may assume $W_k(\tilde t) > 0$ without loss of generality. There is $(t,x) \in E^*[\,\tilde t\,]$ such that

$$\text{(a)} \quad W_k(\tilde t) = u_k(t,x) \qquad \text{or} \qquad \text{(b)} \quad W_k(\tilde t) = -u_k(t,x).$$

Consider the case (a). We conclude that (t,x) is an interior point of E and thus $\partial_x u_k(t,x)=0$, $\partial_{xx}u_k(t,x)\le 0$. We obtain, due to monotonicity of $\tilde\sigma$,

$$\begin{aligned} D_-W_k(\tilde t) &\le D_-W_k(t)\le \partial_t u_k(t,x)=f^{(k)}(t,x,u_{\alpha(t,x)},\partial_x u_k(t,x),\partial_{xx}u_k(t,x))\\ &\le f^{(k)}(t,x,u_{\alpha(t,x)},0,0)\le \tilde\sigma_k(t,Vu_{\alpha(t,x)})\le \tilde\sigma_k(\tilde t,W_{(\tilde t)}). \end{aligned}$$

If we consider the case (b) then $\partial_x u_k(t,x)=0$ and $-\partial_{xx}u_k(t,x)\le 0$. Since $D_-W_k(\tilde t)\le D_-W_k(t)\le -\partial_t u_k(t,x)$, we have

$$\begin{aligned} D_-W_k(\tilde t) &\le -f^{(k)}(t,x,u_{\alpha(t,x)},\partial_x u_k(t,x),\partial_{xx}u_k(t,x))\\ &\le -f^{(k)}(t,x,u_{\alpha(t,x)},0,0)\le \tilde\sigma_k(t,Vu_{\alpha(t,x)})\le \tilde\sigma_k(\tilde t,W_{(\tilde t)}). \end{aligned}$$

The assertion follows from Lemma 2.1. □

We consider again the problem (1.1), (1.2). Let $\tilde\varphi : E^*\to l^\infty$, $\tilde\varphi=\{\tilde\varphi_k\}_{k\in\mathbb{N}}$, be such that $\tilde\varphi(t,x)=\varphi(t,x)$ on $E_0\cup\partial_0E$. Fix $N\in\mathbb{N}$. For $w:E^*\to\mathbb{R}^N$, $w=(w_1,\dots,w_N)$, or $w:E^*\to l^\infty$, $w=\{w_k\}_{k\in\mathbb{N}}$, and for $\tilde w:E^*\to l^\infty$, $\tilde w=\{\tilde w_k\}_{k\in\mathbb{N}}$, put

$$[w,\tilde w]^N=\{\bar w_k\}_{k\in\mathbb{N}},\quad \text{where}\quad \bar w_k=\begin{cases} w_k, & 1\le k\le N,\\ \tilde w_k, & k>N.\end{cases}$$

Consider the differential functional system

$$\partial_t z_k(t,x)=f^{(k)}(t,x,[z,\tilde\varphi]^N_{\alpha(t,x)},\partial_x z_k(t,x),\partial_{xx}z_k(t,x)),\quad 1\le k\le N, \tag{4.1}$$

where $z=(z_1,\dots,z_N)$, with the initial boundary condition

$$z_k(t,x)=\varphi_k(t,x)\quad\text{on}\quad E_0\cup\partial_0E,\quad 1\le k\le N. \tag{4.2}$$

Assumption H $[f,\varphi,\sigma]$**.** The Assumption H $[f,\sigma]$ is satisfied and

1) the function $\varphi\in C(E_0\cup\partial_0E,l^\infty)$ is such that there exists $\tilde\varphi\in C(E^*,l^\infty)$, $\tilde\varphi=\{\tilde\varphi_k\}_{k\in\mathbb{N}}$, with the properties:
   (i) $\tilde\varphi(t,x)=\varphi(t,x)$ for $(t,x)\in E_0\cup\partial_0E$,
   (ii) for each $k\in\mathbb{N}$ the function $\tilde\varphi_k(\cdot,x):[0,a]\to\mathbb{R}$ is of class C^1 and $\tilde\varphi_k(t,\cdot):[-b,b]\to\mathbb{R}$ is of class C^2, $x\in[-b,b]$, $t\in[0,a]$,
   (iii) there is $d\in\mathbb{R}_+$ such that for each $k\in\mathbb{N}$
   $$|\partial_{x_ix_j}\tilde\varphi_k(t,x)|\le d\quad\text{on}\quad E,\quad 1\le i,j\le n,$$
2) for each $k\in\mathbb{N}$ there is $C_k\in\mathbb{R}_+$ such that for $(t,x)\in E$
   $$\Big|f^{(k)}(t,x,\tilde\varphi_{\alpha(t,x)},\partial_x\tilde\varphi_k(t,x),\partial_{xx}\tilde\varphi_k(t,x))-\partial_t\tilde\varphi_k(t,x)\Big|\le C_k$$
   and $\lim_{k\to\infty}C_k=0$.

Remark 4.1. *If we assume that for each $k \in \mathbb{N}$ there are $\tilde{A}_k$, $\tilde{B}_k \in \mathbb{R}_+$ such that for $(t,x) \in E$*

$$|f^{(k)}(t,x,\tilde{\varphi}_{\alpha(t,x)},\partial_x\tilde{\varphi}_k(t,x),\partial_{xx}\tilde{\varphi}_k(t,x))| \le \tilde{A}_k, \qquad |\partial_t\tilde{\varphi}_k(t,x)| \le \tilde{B}_k$$

and $\lim_{k\to\infty}\tilde{A}_k = \lim_{k\to\infty}\tilde{B}_k = 0$, then the condition 2) of Assumption $H\,[f,\varphi,\sigma]$ is satisfied.

Lemma 4.2. *If Assumption $H\,[f,\varphi,\sigma]$ is satisfied and the function $v : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, is a classical parabolic solution of (1.1), (1.2), then for each $k \in \mathbb{N}$ exists $\tilde{\omega}_k \in C(I,\mathbb{R}_+)$ such that*

$$\|v_k - \tilde{\varphi}_k\|_{[\,t\,]} \le \tilde{\omega}_k(t), \quad (t,x) \in E,$$

and $\lim_{k\to\infty}\tilde{\omega}_k(t) = 0$ uniformly on I.

Proof. Define $\tilde{v} : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, by $\tilde{v} = v - \tilde{\varphi}$ on E^*. With this notation, we have $\tilde{v}(t,x) = 0$ on $E_0 \cup \partial_0 E$ and

$$\partial_t\tilde{v}(t,x) = F^{(k)}[\tilde{v}+\tilde{\varphi}](t,x) - \partial_t\tilde{\varphi}_k(t,x), \quad (t,x) \in E,\ k \in \mathbb{N}.$$

Let the function $\{G_k\}_{k\in\mathbb{N}} = G$ be defined on Ω by

$$G_k(t,x,w,q,r) = f^{(k)}(t,x,w+\tilde{\varphi}_{\alpha(t,x)},q+\partial_x\tilde{\varphi}_k(t,x),r+\partial_{xx}\tilde{\varphi}_k(t,x)) - \partial_t\tilde{\varphi}_k(t,x)$$

for $k \in \mathbb{N}$. It satisfies the estimate

$$\begin{aligned}|G_k(t,x,w,0,0)| \;\le\;& |f^{(k)}(t,x,w+\tilde{\varphi}_{\alpha(t,x)},\partial_x\tilde{\varphi}_k(t,x),\partial_{xx}\tilde{\varphi}_k(t,x)) \\ &\quad -f^{(k)}(t,x,\tilde{\varphi}_{\alpha(t,x)},\partial_x\tilde{\varphi}_k(t,x),\partial_{xx}\tilde{\varphi}_k(t,x))| \\ &+|f^{(k)}(t,x,\tilde{\varphi}_{\alpha(t,x)},\partial_x\tilde{\varphi}_k(t,x),\partial_{xx}\tilde{\varphi}_k(t,x)) - \partial_t\tilde{\varphi}_k(t,x)| \\ \le\;& \sigma_k(t,Vw) + C_k\end{aligned}$$

for $(t,x,w) \in E \times C(D,l^\infty)$ and $k \in \mathbb{N}$. It is easy to see that $\tilde{v}$ is a parabolic solution of the mixed problem

$$\begin{aligned}\partial_t z_k(t,x) &= G_k(t,x,z_{\alpha(t,x)},\partial_x z_k(t,x),\partial_{xx}z_k(t,x)) \quad &&\text{on} \quad E, \quad k \in \mathbb{N},\\ z_k(t,x) &= 0 &&\text{on} \quad E_0 \cup \partial_0 E, \quad k \in \mathbb{N}.\end{aligned}$$

By Assumption H $[f,\varphi,\sigma]$, the function $\tilde{\sigma} = \sigma + C$ fulfils, together with $\tilde{v}$, the conditions of Lemma 4.1, and hence

$$|\tilde{v}_k(t,x)| \le \tilde{\omega}_k(t) \quad \text{on} \quad E, \quad k \in \mathbb{N},$$

where $\{\tilde{\omega}_k\}_{k\in\mathbb{N}} = \tilde{\omega}$ is the maximal solution of the problem

$$\omega'(t) = \sigma(t,\omega_{(t)}) + C, \qquad \omega_{(0)} \equiv 0.$$

Since $\lim_{k\to\infty} C_k = 0$, the assertion follows from the condition 4) of Assumption H $[f,\sigma]$. □

Lemma 4.3. *Suppose that Assumption H $[f, \varphi, \sigma]$ is satisfied and the function $v : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, is a classical parabolic solution of* (1.1), (1.2) *and*

1) *there is $c_0 \in \mathbb{R}_+$ such that*

$$|\partial_{x_i x_j} v_k(t,x)| \le c_0 \quad \text{on} \quad E, \quad 1 \le i,j \le n, \quad k \in \mathbb{N},$$

2) *for each $N \in \mathbb{N}$ the function $u^{[N]} : E^* \to \mathbb{R}^N$, $u^{[N]} = (u_1^{[N]}, \dots, u_N^{[N]})$, is a solution of* (4.1), (4.2).

Then for each $N \in \mathbb{N}$ there is $\omega^{[N]} \in C(I, \mathbb{R}_+)$ such that

$$\|v_k - u_k^{[N]}\|_{[\,t\,]} \le \omega^{[N]}(t), \quad t \in [0,a), \quad 1 \le k \le N,$$

and $\lim_{N\to\infty} \omega^{[N]}(t) = 0$ uniformly on I.

Proof. Let us fix $N \in \mathbb{N}$. We introduce the function $v^{[N]} : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, defined as

$$v_k^{[N]} = [v - u^{[N]}, 0]^N \quad \text{on} \quad E^*.$$

With this notation, we have

$$\begin{aligned}\partial_t v_k^{[N]}(t,x) = f^{(k)}(t,x,v_{\alpha(t,x)}, \partial_x v_k(t,x), \partial_{xx} v_k(t,x)),\\ - f^{(k)}(t,x,[u^{[N]},\tilde{\varphi}]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x))\end{aligned}$$

for $1 \le k \le N$, $(t,x) \in E$. Let $H^{[N]} : \Omega \to l^\infty$, $H = \{H_k\}_{k\in\mathbb{N}}$, be defined on Ω by

$$H_k^{[N]} = \begin{cases} \widetilde{H}_k, & 1 \le k \le N, \\ 0, & k > N, \end{cases}$$

where

$$\begin{aligned}&\widetilde{H}_k^{[N]}(t,x,w,q,r)\\ &\quad = f^{(k)}(t,x,w + [u^{[N]},v]^N_{\alpha(t,x)}, q + \partial_x u_k^{[N]}(t,x), r + \partial_{xx} u_k^{[N]}(t,x))\\ &\qquad - f^{(k)}(t,x,[u^{[N]},\tilde{\varphi}]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x)), \quad 1 \le k \le N.\end{aligned}$$

For $1 \le k \le N$, it satisfies the estimate

$$\begin{aligned}|H_k^{[N]}(t,x,w,0,0)| \le \Big| & f^{(k)}(t,x,w + [u^{[N]},v]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x))\\ & - f^{(k)}(t,x,[u^{[N]},v]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x))\Big|\\ + \Big| & f^{(k)}(t,x,[u^{[N]},v]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x))\\ & - f^{(k)}(t,x,[u^{[N]},\tilde{\varphi}]^N_{\alpha(t,x)}, \partial_x u_k^{[N]}(t,x), \partial_{xx} u_k^{[N]}(t,x))\Big|,\end{aligned}$$

which implies

$$|H_k^{[N]}(t,x,w,0,0)| \le \sigma_k(t,Vw) + \sigma_k(t,V[0,v-\tilde{\varphi}]^N_{\alpha(t,x)}).$$

By Lemma 4.2, for $(t,x)\in E$

$$V(v_k-\tilde{\varphi}_k)_{\alpha(t,x)} \le \lim_{\tau\to a^-}\tilde{\omega}_k(\tau)=\varepsilon_k, \quad k\in\mathbb{N}, \qquad \text{and} \quad \lim_{k\to\infty}\varepsilon_k=0.$$

Put $\varepsilon^{[N]}=\{\varepsilon_k^{[N]}\}_{k\in\mathbb{N}}$, where

$$\varepsilon_k^{[N]}=\begin{cases}0, & 1\le k\le N,\\ \varepsilon_k, & k>N.\end{cases}$$

Clearly, $\lim_{N\to\infty}\|\varepsilon^{[N]}\|_\infty=0$. Hence, and by Remark 3.1, for each $N\in\mathbb{N}$ there is $C^{[N]}\in\mathbb{R}_+$ such that for $k\in\mathbb{N}$, $(t,x)\in E$

$$\sigma_k(t,V[0,v-\tilde{\varphi}]^N_{\alpha(t,x)}) \le \sigma_k(t,\varepsilon^{[N]}) \le C^{[N]}$$

and $\lim_{N\to\infty}C^{[N]}=0$. Moreover, $v^{[N]}$ is a parabolic solution of the mixed problem

$$\begin{aligned}\partial_t z_k(t,x) &= H_k^{[N]}(t,x,z_{\alpha(t,x)},\partial_x z_k(t,x),\partial_{xx}z_k(t,x)) && \text{on} \quad E, \quad k\in\mathbb{N},\\ z_k(t,x) &= 0 && \text{on} \quad E_0\cup\partial_0 E, \quad k\in\mathbb{N}.\end{aligned}$$

By Assumption H $[f,\varphi,\sigma]$, the function $\tilde{\sigma}=\{\sigma_k+C^{[N]}\}_{k\in\mathbb{N}}$ fulfils, together with $v^{[N]}$, the conditions of Lemma 4.1, and hence

$$\|v_k^{[N]}\|_{[t]} \le \omega_k^{[N]}(t) \quad \text{on} \quad E, \quad k\in\mathbb{N},$$

where $\{\omega_k^{[N]}\}_{k\in\mathbb{N}}$ is a solution of the problem

$$\begin{aligned}&\eta_k'(t)=\sigma_k(t,\eta_{(t)})+C^{[N]}, \quad k\in\mathbb{N},\\ &\eta_{(0)}\equiv 0.\end{aligned}$$

Since $\lim_{N\to\infty}C^{[N]}=0$, the assertion follows, for

$$\omega^{[N]}(t)=\max\left\{\omega_k^{[N]}(t): 1\le k\le N\right\}, \quad t\in I,$$

from the condition 5) of Assumption H $[f,\sigma]$ and from Lemma 2.1. □

Let $\varphi_h : E_{0.h}\cup\partial_0 E_h \to l^\infty$, $\varphi_h=\{\varphi_{h.k}\}_{k\in\mathbb{N}}$, be given. Consider the difference problem

$$\delta_0 z_k^{(r,m)} = f^{(k)}(t^{(r)},x^{(m)},[T_h z,\tilde{\varphi}]^N_{\alpha(t,x)},\delta z_k(t^{(r)},x^{(m)}),\delta^{(2)}z_k(t^{(r)},x^{(m)})), \tag{4.3}$$

$$z_k^{(r,m)}=\varphi_{h.k}^{(r,m)} \quad \text{on} \quad E_{0.h}\cup\partial_0 E_h, \tag{4.4}$$

for $1\le k\le N$, where $z=(z_1,\dots,z_N)$. We are ready to prove the main theorem in this part of the paper.

Theorem 4.1. *Suppose that Assumptions $H\,[f,H]$, $H\,[f,\varphi,\sigma]$ are satisfied and*

1) *derivatives of f with respect to r and q are bounded,*
2) *the function $v : E^* \to l^\infty$, $v = \{v_k\}_{k\in\mathbb{N}}$, is a parabolic classical solution of (1.1), (1.2),*
3) *the derivatives $\partial_t v_k$, $\partial_{xx} v_k$, $k \in \mathbb{N}$, are equicontinuous,*
4) *for each $N \in \mathbb{N}$ the function $u^{[N]} : E^* \to \mathbb{R}^N$, $u^{[N]} = (u_1^{[N]}, \ldots, u_N^{[N]})$, is a solution of (4.1), (4.2), and the constant $c_k \in \mathbb{R}_+$ is such that $|\partial_{x_i x_j} u_k^{[N]}(t,x)| \le c_k$ on E, $1 \le i,j \le n$, $1 \le k \le N$,*
5) *for each $N \in \mathbb{N}$, $h \in H$ the function $u_h^{[N]} : E_h^* \to \mathbb{R}^N$, $u^{[N]} = (u_{h.1}^{[N]}, \ldots, u_{h.N}^{[N]})$, is a solution of (4.3), (4.4),*
6) *initial-boundary data φ_h satisfy the condition 3) of the Theorem 3.2.*

Then there is $\tilde{\gamma} : H \to \mathbb{R}_+$ such that, for any $N \in \mathbb{N}$,

$$\varepsilon_{\max}(h,N) = \max_{\substack{(t^{(r)},x^{(m)})\in E_h \\ 1\le k\le N}} |(u_{h.k}^{[N]})^{(r,m)} - v_k(t^{(r)},x^{(m)})| \le \tilde{\gamma}(h) + \varepsilon^{[N]}, \tag{4.5}$$

and $\lim\limits_{h\to 0} \tilde{\gamma}(h) = 0$ and $\lim_{N\to\infty} \varepsilon^{[N]} = 0$.

Proof. Let us fix $N \in \mathbb{N}$. Using the method from the proof of Theorem 3.2, we can prove that

$$\left|(u_{h.k}^{[N]})^{(r,m)} - u_k^{[N]}(t^{(r)},x^{(m)})\right| \le \tilde{\gamma}(h) \quad \text{on} \quad E_h, \quad 1 \le k \le N,$$

where $\lim\limits_{h\to 0} \tilde{\gamma}(h) = 0$. Uniformity of this estimate, with respect to N, follows from the relevant uniformity of estimates $\beta^{(0)}(h)$, $\beta^{(1)}(h)$ in the above mentioned proof.

Moreover, from Lemma 4.3,

$$|v_k(t^{(r)},x^{(m)}) - u_k^{[N]}(t^{(r)},x^{(m)})| \le \omega^{[N]}(t^{(r)}) \quad \text{on} \quad E_h, \quad 1 \le k \le N.$$

Thus we obtain the assertion (4.5) with $\varepsilon^{[N]} = \lim_{t\to a^-} \omega^{[N]}(t)$. □

Remark 4.2. *We may choose σ to be linear: $\sigma(t,w) = Aw(\tau)$, where $A \in \mathcal{L}(l_0^\infty) \cap \mathcal{L}(l^\infty)$ and $\tau \in I_0$, and l_0^∞ is the space of those sequences of nonnegative real numbers, which converge to zero. The σ, so chosen, fulfils all the preceding Assumptions, as follows from the thoery of linear differential inequalities in Banach spaces (see, for example, §72 in [11]). Sufficient condition for an inifinite rank matrix $A = [p_{ij}]$, $p_{ij} \in \mathbb{R}_+$, to be in $\mathcal{L}(l_0^\infty) \cap \mathcal{L}(l^\infty)$ is to have a diagonal such that row sums starting from the diagonal are uniformly bounded, while partial row sums to the left of diagonal are vanishing with the row number:*

$$\exists_{n_0} \quad \exists_{M\in\mathbb{R}_+} \forall_i \sum_{j=i+n_0}^{\infty} p_{ij} \le M \quad \textit{and} \quad \lim_{i\to\infty} \sum_{j=1}^{i+n_0-1} p_{ij} = 0.$$

Example 4.1. *Suppose that $F : E \times l^\infty \times \mathbb{R}^n \times M_{n\times n} \to l^\infty$ is given. Then, for $f_k(t,x,w,q,r) = F_k(t,x,w(0,0),q,r)$, $k \in \mathbb{N}$, the equation* (1.1) *reduces to the infinite system with deviating variables*

$$\partial_t z_k(t,x) = F_k(t,x,z(\alpha(t,x)), \partial_x z_k(t,x), \partial_{xx} z_k(t,x)), \quad k \in \mathbb{N},$$

where $z(\alpha(t,x)) = \{z_k(\alpha_k(t,x))\}_{k\in\mathbb{N}}$.

Example 4.2. *Suppose that the function F is given, as in the previous Example. Then, for $f_k(t,x,w,q,r) = F_k(t,x,Aw,q,r)$, where $(Aw)_k = \int_D w_k(s,y)\,ds\,dy$, $k \in \mathbb{N}$, the equation* (1.1) *reduces to the infinite system of integro-differential equations*

$$\partial_t z_k(t,x) = F_k(t,x, \int_{\alpha(t,x)+D} z, \partial_x z_k(t,x), \partial_{xx} z_k(t,x)), \quad k \in \mathbb{N},$$

where

$$\int_{\alpha(t,x)+D} z = \left\{ \int_{\alpha_k(t,x)+D} z_k(s,y)\,ds\,dy \right\}_{k\in\mathbb{N}}.$$

5. Numerical examples

Example 5.1. Let $n = 2$, $a \le 1$ and $E = [0,a] \times (-1,1)^2$, $E_0 = \{0\} \times [-1,1]^2$, $\partial_0 E = [0,a] \times \Big([-1,1]^2 \setminus (-1,1)^2\Big)$. Consider the mixed problem

$$\partial_t z_k(t,x) = f_k(t,x,z_{\alpha(t,x)}, \partial_x z_k(t,x), \partial_{xx} z_k(t,x)), \qquad k \in \mathbb{N}, \tag{5.1}$$

$$z_k(t,x) = k^{-5} \quad \text{on} \quad E_0 \cup \partial_0 E, \qquad k \in \mathbb{N}, \tag{5.2}$$

where, for $k \in \mathbb{N}$, $\alpha_k(t,x) = (t, x_2, x_1)$,

$$\begin{aligned} f_k(t,x,w,q,r) = \arctan\Big(&r_{11} + r_{22} - \bar{g}_k(t,x) w_k(0,0)\Big), \\ &+ (x_1^2 - 1)(x_2^2 - 1) w_k(0,0) + g_k(w(0,0)), \end{aligned}$$

$$\bar{g}_k(t,x) = 4t^2 x_1^2 (x_2^2 - 1)^2 + 4t^2 x_2^2 (x_1^2 - 1)^2 + 2t x_1^2 + 2t x_2^2 - 4t,$$

$$g_k(p) = \begin{cases} 0, & k = 1, \\ p_{k+1} + p_{k-1} - 2k^6\, \frac{k^4 + 10k^2 + 5}{(k^2-1)^5}\, p_k, & k > 1. \end{cases}$$

The exact solution is $z_k(t,x) = k^{-5} \exp[t(x_1^2 - 1)(x_2^2 - 1)]$; we take $\tilde{\varphi}_k(t,x) = k^{-5}$ on E^*. The following table shows the values of the maximal error $\varepsilon_{\max}(h,N)$ (see (4.5)), for $a = 1/4$ and for chosen h and N.

N	$-\log_2 h_0$	$-\log_2 h_1$	$\tilde{\gamma}(h) + \varepsilon^{[N]}$	$-\log_2(\tilde{\gamma}(h) + \varepsilon^{[N]})$
4	7	2	$3.075600 \cdot 10^{-4}$	11.666845
8	9	3	$7.367134 \cdot 10^{-5}$	13.728537
16	11	4	$1.823902 \cdot 10^{-5}$	15.742612
32	13	5	$1.060963 \cdot 10^{-5}$	16.524267

Example 5.2. Let $n = 1$, $a \leq 1$ and $E = [0,a] \times (-1,1)$, $E_0 = \{0\} \times [-2,2]$, $\partial_0 E = [0,a] \times \Big([-2,-1] \cup [1,2]\Big)$. Consider the mixed problem

$$\partial_t z_k(t,x) = f_k(t,x,z_{\alpha(t,x)}, \partial_x z_k(t,x), \partial_{xx} z_k(t,x)), \qquad k \in \mathbb{N}, \tag{5.3}$$

$$z_k(t,x) = 0 \quad \text{on} \quad E_0 \cup \partial_0 E, \qquad k \in \mathbb{N}, \tag{5.4}$$

where, for $k \in \mathbb{N}$, $\alpha_k(t,x) = (t,-x)$,

$$f_k(t,x,w,q,r) = \arctan\Big(r - \sum_{n=2}^{k+1} 4nta_n(t)b_{n-1}(x)[(4n+1)x^2 - 3]\Big) + \int_{\frac{-x-1}{2}}^{\frac{-x+1}{2}} (w_{k+1} - w_k)(0,s)\,ds + g_k(t,x),$$

$$g_k(t,x) = -\frac{ta_{k+2}(t)}{2(2k+5)} \Big(\frac{3}{4}\Big)^{2k+5} \big(\gamma^{2k+5}(x) - \beta^{2k+5}(x)\big) + \sum_{n=2}^{k+1} 2na_n(t)b_n(x),$$

$$\beta(x) = \begin{cases} 3x^2 - 1 + 2x, & x \in [-1, \frac{1}{3}] \\ 0, & x \in (\frac{1}{3}, 1] \end{cases}, \quad \gamma(x) = \begin{cases} 3x^2 - 1 - 2x, & x \in [-\frac{1}{3}, 1] \\ 0, & x \in [-1, -\frac{1}{3}) \end{cases},$$

$$a_n(t) = (-1)^n \frac{4^n - 4}{(2n)!} t^{2n-1}, \qquad b_n(x) = x(x^2-1)^{2n}, \qquad n \geq 2.$$

The exact solution is $z_k(t,x) = \sum_{n=2}^{k+1} ta_n(t)b_n(x)$, $k \in \mathbb{N}$; we take, for $k \in \mathbb{N}$,

$$\tilde{\varphi}_k(t,x) = \begin{cases} 8x \sin^4(\frac{1}{2}t(x^2-1)) & \text{on} \quad E, \\ 0 & \text{on} \quad E_0 \cup \partial_0 E. \end{cases}$$

The following table shows the values of the maximal error $\varepsilon_{\max}(h,N)$ (see (4.5)), for $a = 1/4$ and for chosen h and N.

N	$-\log_2 h_0$	$-\log_2 h_1$	$\tilde{\gamma}(h) + \varepsilon^{[N]}$	$-\log_2(\tilde{\gamma}(h) + \varepsilon^{[N]})$
4	6	2	$3.607181 \cdot 10^{-5}$	14.758768
8	8	3	$8.435167 \cdot 10^{-6}$	16.855152
16	10	4	$2.084722 \cdot 10^{-6}$	18.871714
32	12	5	$5.218899 \cdot 10^{-7}$	20.869751

Example 5.3. Our result seems to be new also in the classical case, that is, without the functional dependence. An interesting problem, arising in applications (the discrete coagulation-fragmentation model which describes the dynamics of cluster growth and arises in polymer science, atmospheric physics, and colloidal chemistry), is considered in the work [12], in the following form (a sum over an empty range is meant to be zero):

$$\frac{\partial u_k}{\partial t} = d_k \Delta u_k + \frac{1}{2} \sum_{j=1}^{k-1} (a_{k-j,j} u_{k-j} u_j - b_{k-j,j} u_k) - u_k \sum_{j=1}^{\infty} a_{kj} u_j + \sum_{j=1}^{\infty} b_{ij} u_{k+j},$$

$k = 1, 2, \ldots$, on $\Omega_a = (0, a) \times \Omega$, $\Omega \subset \mathbb{R}^n$, with an initial condition and homogeneous Neumann condition. As mentioned in [12], for this problem a Galerkin numerical approximation were considered; we apply a finite difference approximation, presented here.

We choose $n = 1$, $\Omega = (-2, 2)$, and the coefficients: $d_k = 1$ and $a_{kj} = b_{kj} = q^{k+j}$, $q \in (0, 1)$, for all k and j. Testing the difference scheme is done against the known solution, namely $\tilde{v}_k = q^{-k}(v_{k-1} - v_k)$, with $v_k(t, x) = p^k v^4(s^k x + \alpha \sin(t/(k+1)))$ and v being a cosine, extended by zero outside $[-\pi/2, \pi/2]$. The constants are: $q = 27/32$, $p = 1/4$, $s = 3/2$, $\alpha = 3/10$. Such a system satisfies conditions assumed in [12] for global existence. Specifically, the solution components are nonnegative, and the overall mass, that is, the sum of integrals of $k \cdot \tilde{v}_k$ over space domain Ω, is constant in time.

Thus our exemplific problem, which we have made inhomogeneous (F_k) for sake of simplicity, yields

$$\frac{\partial u_k}{\partial t} = \Delta u_k + \frac{1}{2} q^k \sum_{j=1}^{k-1} (u_{k-j} u_j - u_k) - q^k u_k \sum_{j=1}^{\infty} q^j u_j + q^k \sum_{j=1}^{\infty} q^j u_{k+j} - F_k,$$

$k = 1, 2, 3, \ldots$, and $u_k(0, x) = q^{-k} p^{k-1} (v^4(s^{k-1} x) - p v^4(s^k x))$, $u_k(t, -2) = u_k(t, 2) = 0$. Provided that solutions u_k are uniformly bounded (which is a fairly reasonable assumption, and possible to check with the aid of partial differential inequalities), respective right-hand side is Lipschitz continuous with respect to u, and the Lipschitz coefficients form a matrix of infinite rank, in a way exactly as stated in the Remark 4.2. The term F_k depends only on k, t and x, and it is so chosen, that the above mentioned solution is valid.

The following table shows the values of the maximal error $\varepsilon_{\max}(h, N)$ (see (4.5)), for $a = 1/4$, for various sizes of the finite system, and for various step sizes; the last ones are controlled by the choice of $h_1 = 2/N_1$ and by the CFL condition (3.19), in which equality holds. In calculations, we have used the extension $\tilde{\varphi}$ of data, given by $\tilde{\varphi}_k(t, x) = u_k(0, x)$; then $\sum_{j=N+1}^{\infty} q^j u_j = v_N(0, x)$, so that the right-hand side of (4.3) may be effectively computed.

$\log_2(N)$	$\log_2(N_1)$	err_{max}	$-\log_2(err_{max})$
2	4	0.00352021	8.15012
3	4	0.00950878	6.71652
1	4	0.00406853	7.94128
2	5	0.000864932	10.1751
2	6	0.000221877	12.1380
2	7	0.000196772	12.3112
3	7	5.32112e-05	14.1979
4	7	0.000174529	12.4842
3	8	1.33044e-05	16.1977

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