Stochastic Modeling & Applications
Vol. 25 No. 2 (July-December, 2021)
ISSN: 0972-3641

Received: 20th October 2021 Revised:12th December 2021 Accepted: 31st December 2021

BOUND INEQUALITIES FOR MINIMUM DEGREE LAPLACIAN EIGENVALUES OF GRAPHS

SHIVAKUMAR SWAMY C S

Abstract. The Minimum degree energy $E_m(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of the minimum degree matrix $m(G)$. The minimum degree Laplacian matrix of G is defined $L(G) = D(G)-m(G)$, where $D(G)$ is a diagonal matrix of vertex degrees.In this paper we establish some inequalities for minimum degree Laplacian eigenvalues of a graph G.

1. Introduction

The study of spectral graph theory, in essence is concerned with the relationship between the algebraic properties of the spectra of certain matrices associated with a graph, such as the Adjacency matrix, the Distance matrix, the Laplacian matrix and other related matrix.
Let G be a graph(assumed simple throughout) with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges and let d_i be the degree of v_i, $i = 1, 2, 3, \ldots, n$. Define

$$d_{ij} = \begin{cases} min\{d_i, d_j\} & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Then the $n \times n$ matrix $m(G) = (d_{ij})$ is called the minimum degree matrix of G. The characteristic polynomial of the minimum degree matrix $m(G)$ is defined by

$$\begin{aligned}\phi(G; \mu) &= det(\mu I - m(G)) \\ &= \mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_{n-1}\mu + c_n,\end{aligned}$$

where I is the unit matrix of order n. The co-efficient $c_i(i = 0, 1, 2)$ are $c_0 = 1$, c_1 = trace of $m(G) = 0$ and $c_2 = -\sum_{i=1}^{n}(a_i + b_i)d_i^2$, where a_i = the number of vertices in the neighborhood of v_i whose degrees are greater than d_i
and b_i = the number of vertices $v_j(j > i)$ in the neighborhood of v_i whose degrees are equal to d_i.
Note that c_2 and c_2' are negative and so $-c_2 = |c_2|, -c_2' = |c_2'|$.

The minimum degree eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of the graph G are the eigenvalues of its minimum degree matrix $m(G)$. The minimum degree energy of a graph

2000 *Mathematics Subject Classification.* 05C30, 05C50.
Key words and phrases. Minimum degree matrix,minimum degree Laplacian matrix, minimum degree Laplacian eigenvalues of graph.

G is defined as

$$E_m(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1.1}$$

Since m(G) is real symmetric matrix with zero trace, these minimum degree eigenvalues are all real with sum equal to zero.

Equation 1.1 was introduced by the author[13] and was conceived in full analogy to the ordinary graph energy $E(G)$ defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the adjacency matrix of G [4]. The largest eigenvalue λ_1 of the graph G is often called the Spectral radius of G. In literature there are several upper bounds for the spectral radius λ_1(see, e.g.,[9, 3, 15, 14])
Let $D(G)$ be the diagonal matrix of vertex degrees. The minimum degree Laplacian matrix of G is $L(G) = D(G) - m(G)$, where $m(G)$ is the minimum degree matrix of G. Clearly $L(G)$ is a real symmetric matrix. The characteristic polynomial of the minimum degree Laplacian matrix of G is defined by

$$\begin{aligned} \phi(G;\gamma) &= det(\gamma I - L(G)) \\ &= \gamma^n + c_1'\gamma^{n-1} + c_2'\gamma^{n-2} + \cdots + c_{n-1}'\gamma + c_n', \end{aligned}$$

where I is the unit matrix of order n. The eigenvalues of $L(G)$ are called minimum degree Laplacian eigenvalues of G and are denoted by $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. Note that $\sum_{i=1}^{n} \gamma_i = 2m$ and $\sum_{i=1}^{n} \gamma_i^2 = 2\,|c_2'| + \sum_{i=1}^{n} d_i^2$, where d_i is the degree of v_i. In many applications one needs good bounds of the largest laplacian eigenvalues (see for instance, [10, 11, 12]).
In Section 2, we obtain several upper bounds for minimum degree laplacian eigenvalues of G.

2. Bounds for minimum degree laplacian eigenvalues

In this section, we obtain some inequalities involving the minimum degree laplacian eigenvalues of a graph.

Theorem 2.1. *Let G and H be two graphs with n vertices. If $\mu_1, \mu_2, \ldots, \mu_n$ are the minimum degree eigenvalues of G and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the minimum degree Laplacian eigenvalues of H. Then*

$$\sum_{i=1}^{n} \mu_i \gamma_i \leq \sqrt{2\,|c_2| \left(2\,|c_2'| + \sum_{i=1}^{n} d_i^2\right)},$$

where c_2 is the coefficient of μ^{n-2} in the characteristic polynomial of the minimum degree matrix $m(G)$ and c_2' is the coefficient of γ^{n-2} in the characteristic polynomial of the Laplacian matrix of H

Proof. By Cauchy-schwarz inequality, we have

$$\left(\sum_{i=1}^{n}\mu_i\gamma_i\right)^2 \le \left(\sum_{i=1}^{n}\mu_i^2\right)\left(\sum_{i=1}^{n}\gamma_i^2\right)$$
$$\le 2\left|c_2\right|\left(\sum_{i=1}^{n}d_i^2 + 2\left|c_2'\right|\right).$$

Hence,

$$\sum_{i=1}^{n}\mu_i\gamma_i \le \sqrt{2\left|c_2\right|\left(2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right)}. \qquad (2.1)$$

□

Theorem 2.2. *If G is a graph with n vertices and $\gamma_1, \gamma_2, \ldots, \gamma_n$ are the minimum degree Laplacian eigenvalues of G, then*

$$\gamma_1 \le \frac{1}{p-1}\left(\sqrt{p(p-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right]} + \sum_{i=1}^{p}\gamma_{n-p+i}\right), \qquad 2 \le p \le n.$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$, $2 \le p \le n$ be the minimum degree Laplacian eigenvalues of G. Let $H = K_p \bigcup \overline{K_{n-p}}$. The minimum degree eigenvalues of H are $(p-1)^2$, 0(n-p times) and $-(p-1)$ (p-1 times).
Now on employing Theorem 2.1 we obtain,

$$\gamma_1(p-1)^2 - (p-1)\sum_{i=2}^{p}\gamma_{n-p+i} \le \sqrt{p(p-1)^3\left[2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right]}.$$

Hence,

$$\gamma_1 \le \frac{1}{p-1}\left(\sqrt{p(p-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right]} + \sum_{i=2}^{p}\gamma_{n-p+i}\right). \qquad (2.2)$$

Remark 2.1 Setting $p = 2$, in (2.2) we obtain

$$\gamma_1 - \gamma_n \le \sqrt{2\left[2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right]}.$$

□

Corollary 2.1 *If G is a graph with n vertices and m edges, then*

$$\gamma_1 \le \frac{1}{n}\left(\sqrt{n(n-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n}d_i^2\right]} + 2m\right).$$

Proof. If we put $p = n$, in (2.2) we get

$$\gamma_1 \le \frac{1}{n-1}\left(\sqrt{n(n-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]} + \sum_{i=2}^{n}\gamma_i\right).$$

Since

$$\sum_{i=1}^{n}\gamma_i = 2m,$$

we have

$$\gamma_1 \le \frac{1}{n-1}\left(\sqrt{n(n-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]} + (2m - \gamma_1)\right)$$

and hence,

$$\gamma_1 \le \frac{1}{n}\left(\sqrt{n(n-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]} + 2m\right).$$

Theorem 2.3. *Let G be a graph with n vertices. If $\gamma_1 \ge \gamma_2 \ge \cdots \ge \gamma_n$ are the minimum degree Laplacian eigenvalues of G, then*

$$\sum_{i=1}^{k}\gamma_i \le \frac{k}{n}\left(\sqrt{\frac{n(n-k)}{k}\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]} + 2m\right), \qquad 1 \le k \le n.$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_k, \gamma_{k+1}, \ldots, \gamma_n$ be the minimum degree Laplacian eigenvalues of G. Let H be the union of k complete graphs K_p. i.e., $H = \bigcup_k K_p$. The minimum degree eigenvalues of H are $(p-1)^2$ (k times) and $-(p-1)$ [(p-1)k times] and number of vertices and edges of H are $n = pk$ and $\frac{kp(p-1)}{2}$ respectively. Now on employing Theorem 2.1, we get

$$(p-1)^2\gamma_1 + \cdots + (p-1)^2\gamma_k - (p-1)\gamma_{k+1} - \cdots - (p-1)\gamma_n \le \sqrt{kp(p-1)^3\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]}$$

i.e.,

$$p\sum_{i=1}^{k}\gamma_i - \sum_{i=1}^{n}\gamma_i \le \sqrt{kp(p-1)\left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2\right]}.$$

Since

$$n = pk \ \text{ and } \ \sum_{i=1}^{n}\gamma_i = 2m,$$

we have,

$$\sum_{i=1}^{k} \gamma_i \leq \frac{k}{n} \left(\sqrt{\frac{n(n-k)}{k} \left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2 \right]} + 2m \right). \tag{2.3}$$

□

Remark 2.2 Taking $k = 1$, in (2.3) we see that

$$\gamma_1 \leq \frac{1}{n} \left(\sqrt{n(n-1) \left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2 \right]} + 2m \right).$$

Theorem 2.4. *Let G be a graph with n vertices. If $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ are the minimum degree Laplacian eigenvalues of G, then*

$$\sum_{i=1}^{k} [\gamma_i - \gamma_{p-k+i}] \leq \sqrt{k \left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2 \right]}, \qquad 1 \leq k \leq \left[\frac{n}{2} \right].$$

Proof. Let $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_k \geq \gamma_{k+1} \geq \ldots \geq \gamma_{n-k} \geq \gamma_{n-k+1} \geq \ldots \geq \gamma_n$ be the minimum degree Laplacian eigenvalues of G. Let H be a graph with n vertices and k components each is complete bipartite graph $K_{p,q}$ i.e., $H = \bigcup_k K_{p,q}$.

The minimum degree eigenvalues of H are $p\sqrt{pq}$ [k times], 0[(n-2k) times] and $-p\sqrt{pq}$ [k times] and the number of vertices and edges of H are $n = k(p+q)$ and kpq respectively.

On employing Theorem 2.1, we get

$$p\sqrt{pq} \sum_{i=1}^{k} \gamma_i - p\sqrt{pq} \sum_{i=1}^{k} \gamma_{n-k+i} \leq \sqrt{kp^3q \left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2 \right]}.$$

Thus,

$$\sum_{i=1}^{k} [\gamma_i - \gamma_{p-k+i}] \leq \sqrt{k \left[2\left|c_2'\right| + \sum_{i=1}^{n} d_i^2 \right]}. \tag{2.4}$$

□

Acknowledgment. The author is thankful to Prof.Chandrashekara Adiga for his encouragement and suggestions.

References

1. C. Adiga and C. S. Shivakumar Swamy, Bounds on the largest of minimum degree eigenvalues of graphs, *International Mathematical Forum*, **5** (37) (2010), 1823-1831.
2. C. Adiga and C. S. Shivakumar Swamy, Bounds on the largest of minimum degree laplacian eigenvalues of graphs, *International Math. Combin.,* **2** (2011), 130-136.
3. K. C. Das and Pavan Kumar, Bounds on the greatest eigenvalues of graphs, *Indian J.pure appl. Math.*, **34**(6) (2003), 917-925.
4. I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz*, 103(1978), 1-22.
5. I. Gutman, O. E. Polansky, *Mathematical Concepts in Organic Chemistry*, SpringerVerlag, Berlin, 1986.
6. I. Gutman, McClelland-type lower bound for total $\pi-$electron energy, *J. Chem. Soc. Faraday Trans.*, 86 (1990), 3373-3375.
7. I. Gutman, Total $\pi-$electron energy of benzenoid hydrocarbons, *Topics Curr. Chem.,* 162 (1992), 29-63.
8. I. Gutman, The energy of a graph: old and new results, *Combinatorics and applications, A. Betten, A. Khoner,R. Laue and A. Wassermann, eds., Springer, Berlin*, (2001), 196-211.
9. Y. Hong, Bounds of eigenvalues of graphs, *Discreate Mathematics*, **123** (1993), 65-74.
10. R. Merris, Laplacian matrices of graphs: *a survey, Linear Algebra Appl.*, **197-198** (1994), 143-176.
11. R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra Appl.*, **285** (1998), 33-35.
12. B. Mohar, Some applications of Laplace eigenvalues of graphs, *in: G. Hahn, G. Sabidussi(Eds.), Graph Symmetry, Kluwer Academic Publishers, Dordrecht*, (1997), 225-275.
13. Shivakumar Swamy. C. S, On the minimum degree energy of a graph, *Advances and Applications in Discrete Mathematics*, **4**(2) (2009), 137-145.
14. R. P. Stanley, *Linear Algebra Appl.*, **67** (1987), 267-69.
15. H. S. Wilf, The eigenvalues of a graphand its chromatic number, *J.London Math.*, **42** (1967), 330-332.

Department of Mathematics, Government College for Women (Autonomous) Mandya-571401, INDIA.

Email address: `cskswamy@gmail.com`