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SOME FIXED POINT THEOREMS IN GENERATING POLISH SPACE OF QUASI METRIC FAMILY

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ABSTRACT: The generating space of quasi metric family is important because of its involvement with fuzzy and probabilistic metric space.
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1. Introduction and Preliminary

The generating space of quasi 2-metric family is important because of its involvement with fuzzy and probabilistic 2-metric space and a minimization theorem [1], [3] is to obtain fixed point theorem. In 2008 V.B. Dhagat and V.S. Thakur proved non convex minimization theorem for generating space of quasi 2-metric family . In this paper, we prove a minimization theorem for sequence of mappings Ta for $a \in \mathrm{N}$ and further we prove fixed point theorem as an application of minimization theorem with non commuting condition Known as weak compatible.

1.1. Definitions.

1.1.1. Generating Polish Space of Quasi Metric Family. Let X be non empty set and $\{d_\alpha: \alpha \in (0,1]\}$ be a family of mappings d_α of $(\Omega \times X) \times (\Omega \times X)$ into R^+, $\omega \in \Omega$ be a selector. $(X, d_\alpha: \alpha \in (0, 1])$ is called generating Polish space of quasi metric family if it satisfies the following conditions:

1. $d_\alpha((\omega, x), (\omega, y)) = 0 \ \forall \ \alpha \in (0, 1] \Leftrightarrow x = y$
2. $d_\alpha((\omega, x), (\omega, y)) = d_\alpha((\omega, y), (\omega, x)) \ \forall \ x, y \in X, \omega \in \Omega$ and $\alpha \in (0, 1]$
3. For any $\alpha \in (0, 1]$, there exists a number $\mu \in (0, \alpha]$ such that: $d_\alpha((\omega, x), (\omega, y)) = d_\mu((\omega, x), (\omega, z)) + d_\mu((\omega, z), (\omega, y)) \ \forall \ x, y \in X, \omega \in \Omega$ be a selector.
4. For any $x, y \in X$, $\omega \in \Omega$, $d_\alpha((\omega, x), (\omega, y))$ is non-increasing and left continuous in α.

1.1.2. Quasi Compatible. Let $(X, d_\alpha: \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. The mapping S and T are said to be quasi compatible if

$$d_\alpha(ST(\omega, x_n), TS(\omega, x_n)) \to 0 \text{ as } n \to \infty, \alpha \in (0, 1], \omega \in \Omega$$

whenever $\{\omega, x_n\}$ be a sequence in $\Omega \times X$ such that $\lim_{n \to \infty} S(\omega, x_n) = \lim_{n \to \infty} T(\omega, x_n) = p$ for some $p \in X$.

Keyword. Fixed point, Quasi 2-metric family, Generating polish Space.

1.1.3. Compatible of Type (A). Let $(X, d_\alpha: \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. The mapping S and T are said to be compatible of type (A) if:

$$d_\alpha(TS(\omega, x_n), SS(\omega, x_n)) = 0 \text{ and } d_\alpha(ST(\omega, x_n), TT(\omega, x_n))=0$$

whenever $\{\omega, x_n\}$be a sequence in $\Omega \times X$ such that $\lim_{n \to \infty} S(\omega, x_n) = \lim_{n \to \infty} T(\omega, x_n) = p$ for some $p \in X$

1.1.4. Implicit Relation. Let ϕ be the set of all real functions $\phi: R_+^4 \to R$ such that:

(F_1): Φ is continuous in each coordinate variable,

(F_2): If either $\phi(u, 0, u, v) \le 0$ or $\phi(u, 0, u + v, v) \le 0$ for all $u, v \ge 0$, then there exists a real constant $0 \le h \le 1$ such that $u \le hv$.

2. Some Concerning Results

Lemma 2.1: Let $(X, d_\alpha: \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. Suppose that

$$\lim_{n \to \infty} S(\omega, x_n) = \lim_{n \to \infty} T(\omega, x_n) = p \text{ for some } p \in X.$$

Then we have the following

1. $\lim_{n \to \infty} ST(\omega, x_n) = Tp$ if T is continuous and
2. $STp = TSp$ and $Sp = Tp$ if T is continuous

Proof 2.1(1): Suppose that

$$\lim_{n \to \infty} S(\omega, x_n) = \lim_{n \to \infty} T(\omega, x_n) = p \text{ for some } p \in X.$$

Now, since T is continuous, we have

$$\lim_{n \to \infty} TS(\omega, x_n) = Tp$$

By 1.1.1(3), we have

$$d_\alpha(ST(\omega, x_n), Tp) = d_\mu(ST(\omega, x_n), TS(\omega, x_n)) + d_\mu(TS(\omega, x_n), Tp);\ \mu \in (0, \alpha]$$

Since S and T are quasi compatible, we have

$$\lim_{n \to \infty} ST(\omega, x_n) = Tp$$

Proof 2.1(2): Since T is continuous,

$$\lim_{n \to \infty} ST(\omega, x_n) = Tp$$

Hence by uniqueness of limit, we have $Sp = Tp$

Now again $d_\alpha(STp, TSp) = \lim_{n \to \infty} d_\alpha (ST(\omega, x_n), TS(\omega, x_n)) = 0$

i.e. $STp = TSp$

This completes the proof.

Lemma 2.2: Let $(X, d_\alpha: \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S and T be mappings from $\Omega \times X$ into X. If S_i and T_j are weakly compatible for any $\alpha \in (0, 1]$ and for $\mu \in (0, \alpha]$.

Then, $S_iT_jp = T_jT_jp = T_jS_ip = S_iS_ip$

Proof: Suppose $\{\omega, x_n\}$ be a sequence in $\Omega \times X$ defined by $x_n = p$ as $n \to \infty$ and $Sp = Tp$.

Then we have

$$\lim_{n \to \infty} S(\omega, x_n) = \lim_{n \to \infty} T(\omega, x_n) = p$$

Since S and T have weakly compatible), we have

$$d_\alpha(STp, TTp) = \lim_{n \to \infty} d_\alpha(ST(\omega, x_n), TT(\omega, x_n)) = 0; \alpha \in (0, 1]$$

Hence, we have, $STp = TTp$

Similarly, we have, $TSp = SSp$

But, $Tp = Sp$

It implies, $TTp = TSp$

Therefore, $STp = TTp = TSp = SSp$

3. Main Results

Theorem 3.1: Let, $(X, d_\alpha: \alpha \in (0, 1])$ be a Generating Polish space of quasi metric family and S, T & G are mapping from $\Omega \times X \to X$ are continuous random operator w.r.t. d. Suppose there are some $\alpha \in (0, 1]$ such that for $x, y \in X$ and $\xi \in \Omega$, we have the following conditions

1. $S(X) \subseteq G(X)$ and $T(X) \subseteq G(X)$
2. $\phi \begin{Bmatrix} d_\alpha\left(S(\omega, x), T(\omega, y)\right), d_\alpha\left(S(\omega, x), G(\omega, y)\right), \\ d_\alpha\left(G(\omega, x), T(\omega, y)\right), d_\alpha\left(G(\omega, x), G(\omega, y)\right) \end{Bmatrix} \leq 0$

   $\forall x, y \in X$ and $\alpha \in (0, 1]$,

(3) G is continuous

(4) The pairs {S, G} and {T, G} are weakly compatible on X.

Then S, T and G have common fixed point.

Proof:

Let, x_0 be any arbitrary point of X.

Since, $S(X) \subseteq G(X)$ and $T(X) \subseteq G(X)$

and $SG(X) \subseteq GG(X)$ and $TG(X) \subseteq GG(X)$

So there exists x_1 and x_2 in X such that

$$GG(\omega, x_1) = SG(\omega, x_0) \text{ and } GG(\omega, x_2) = TG(\omega, x_1)$$

In general

$$GG(\omega, x_{2n+1}) = SG(\omega, x_{2n}) \text{ and } GG(\omega, x_{2n+2}) = TG(\omega, x_{2n+1})$$

for, $n = 0, 1, 2, 3, \ldots$

Let, $d_n = d_\alpha(GG(\omega, x_n), GG(\omega, x_{n+1}))$

Also we know

$$d_\alpha(GG(\omega, x_{2n}), GG(\omega, x_{2n+2})) \leq \begin{Bmatrix} d_\mu(GG(\omega, x_{2n}), GG(\omega, x_{2n+1})) \\ + d_\mu(GG(\omega, x_{2n+1}), GG(\omega, x_{2n+2})) \end{Bmatrix}$$

$\forall x, y \in X$ and $\mu \in (0, \alpha]$, Suppose x_{2n}, x_{2n+1} satisfy 3.1(2) then $\forall\alpha \in (0, 1]$

$$\phi\begin{Bmatrix} d_\alpha\left(SG\left(\omega, x_{2n}\right), TG\left(\omega, x_{2n+1}\right)\right), d_\alpha\left(SG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+1}\right)\right), \\ d_\alpha\left(GG\left(\omega, x_{2n}\right), TG\left(\omega, x_{2n+1}\right)\right), d_\alpha\left(GG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+1}\right)\right) \end{Bmatrix} \le 0$$

$$\phi\begin{Bmatrix} d_\alpha\left(GG\left(\omega, x_{2n+1}\right), GG\left(\omega, x_{2n+2}\right)\right), d_\alpha\left(GG\left(\omega, x_{2n+1}\right), GG\left(\omega, x_{2n+1}\right)\right), \\ d_\alpha\left(GG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+2}\right)\right), d_\alpha\left(GG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+1}\right)\right) \end{Bmatrix} \le 0$$

$$\phi\begin{Bmatrix} d_\alpha\left(GG\left(\omega, x_{2n+1}\right), GG\left(\omega, x_{2n+2}\right)\right), 0, \left[d_\mu\left(GG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+1}\right)\right)\right. \\ \left. + d_\mu\left(GG\left(\omega, x_{2n+1}\right), GG\left(\omega, x_{2n+2}\right)\right)\right], d_\alpha\left(GG\left(\omega, x_{2n}\right), GG\left(\omega, x_{2n+1}\right)\right) \end{Bmatrix} \le 0$$

Thus by definition of implicit relation 1.1.3 we have

$$d_\alpha(GG(\omega, x_{2n+1}), GG(\omega, x_{2n+2})) \le h\{d_\alpha(GG(\omega, x_{2n}), GG(\omega, x_{2n+1}))\}$$

$$d_{2n+1} \le hd_{2n}$$

$$d_{2n+1} \le d_{2n}$$

Similarly, $$d_{2n} \le hd_{2n+1}$$

Thus, $\{d_{2n}\}$ be monotone decreasing and hence converge to zero.

Therefore, $\{GG(\omega, x_{2n})\}$ is a Cauchy sequence and converge to Gp and hence to point X.

Since $\{SG(\omega, x_{2n})\}$ and $\{TG(\omega, x_{2n})\}$ are subsequence of $\{GG(\omega, x_{2n})\}$ and so converge to same point p.

Now by lemma 2.1 we obtain

$$SGp = GSp \text{ and } Sp = Gp$$

Similarly, $$TGp = GTp \text{ and } Tp = Gp$$

Hence, $$Sp = Tp = Gp$$

Also, $$Sp = p = Gp = Tp \text{ as } Gp = p$$

Hence, p is common fixed point of S, T and G.

This completes the proof.

Corollary 3.2: Let, $(X, d_\alpha: \alpha \in (0, 1])$ be a generating Polish space of quasi metric family and S, T and G be mappings from $\Omega \times X$ into X satisfying

1. $S(X) \subseteq G(X)$ and $T(X) \subseteq G(X)$
2. $$\phi\begin{Bmatrix} d_\alpha\left(S\left(\omega, x\right), T\left(\omega, y\right)\right), d_\alpha\left(S\left(\omega, x\right), G\left(\omega, y\right)\right), \\ d_\alpha\left(G\left(\omega, x\right), T\left(\omega, y\right)\right), d_\alpha\left(G\left(\omega, x\right), G\left(\omega, y\right)\right) \end{Bmatrix} \le 0$$

   $\forall x, y \in X$ and $\alpha \in (0, 1]$,
3. G is continuous
4. The pairs {S, G} and {T, G} are weakly compatible Then S, T and G have common fixed point.

Proof: Similar to the proof of the theorem 3.1 using the fact that Quasi compatible pair of maps is weakly compatible but converse is not always true.

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