Global and Stochastic Analysis
Vol. 12 No. 2 (March, 2025)

Received: 11th January 2025 *Revised: 24th January 2025* *Accepted: 25th January 2025*

INVERSE BLOCK DOMINATION AND RELATED PARAMETERS IN GRAPHS

SUREKHA RAVISHANKAR BHAT, R. S. BHAT, AND SMITHA GANESH BHAT*

ABSTRACT. The concept of inverse domination was introduced by V.R. Kulli and S.C. Sigarakanti. Given a graph G, let D represent a γ-set of G. A dominating set $D_1 \subseteq V - D$ is termed an inverse dominating set of G with respect to D. The inverse domination number, denoted by $\gamma'(G)$, is the cardinality of the smallest inverse dominating set. Although inverse domination has been widely explored, the literature provides relatively few bounds on this parameter. Several bounds have been established in terms of graph parameters such as order, size, maximum degree, and domatic number. Additionally, various inverse block domination parameters have been introduced, with initial studies examining their properties. In this paper, we derive an upper bound for the inverse domination number of a graph in terms of its domatic number. Furthermore, a lower bound is provided in terms of the graph's order and size.

1. Introduction

For any undefined terminology, we refer to the works of Harary [4] and West [11]. A graph, in this context, refers to a connected, finite, simple graph with p vertices and q edges. A vertex $v \in V$ is called a *cut-vertex* of a graph G if the removal of v, denoted $G-v$, results in a disconnected graph. An edge is referred to as a *bridge* or *cut-edge* if its removal disconnects the graph. A graph G is *separable* if it contains at least one cut-vertex, otherwise it is called *nonseparable*. A maximal nonseparable subgraph of G is termed a *block* of G. Let $B(G)$ and $C(G)$ denote the set of all blocks and cut-vertices of G repectively. Consider $|V(G)| = p$ called the order and $|E(G)| = q$ called the size of the graph. While $|B(G)| = m$ and $|C(G)| = n$. If a block $b \in B(G)$ contains a cut-vertex $c \in C(G)$ then we say that b and c incident to each other. Two blocks in G are adjacent if there is a common cut-vertex incident on them. On the other hand two cut-vertices are adjacent if there is a common block incident on them. A $block-graph$ $B_G(G)$ is a graph with vertex set $B(G)$ and any two blocks b_1 and b_2 are adjacent in $B_G(G)$ if and only if there is a common cut-vertex incident on b_1 and b_2. A $cut-vertex-graph$ $C_G(G)$ is a graph with vertex set $C(G)$ and any two cut-vertices c_1 and c_2 are adjacent in $C_G(G)$ if and only if there is a common block incident on c_1 and c_2. It is observed that $B_G(B_G(G)) = C_G(G)$. Further a *block cut vertex graph* $BC(G)$ is a tree

2000 *Mathematics Subject Classification.* 05C69.

Key words and phrases. Domination number, Inverse domination number, domatic number, inverse vertex-block domination number, inverse block-vertex domination number.

*Corresponding Author - Smitha Ganesh Bhat, e-mail- smitha.holla@manipal.edu.

with vertex set $B(G) \cup C(G)$ and a cut-vertex $c \in C(G)$ and a block $b \in B(G)$ are adjacent in $BC(G)$ if and only if c is incident with the block b. A *block vertex tree* $b_p(G)$ is a tree with vertex set $B(G) \cup V(G)$ and a vertex $v \in V(G)$ and a block $b \in B(G)$ are adjacent in $b_p(G)$ if and only if v is incident on the block b. A block b is a *pendant block* if it is incident with only one cut-vertex, otherwise b is a non-pendant block.

The study of mixed domination was initiated by E. Sampathkumar and S. S. Kamath [8]. For any graph G, the vertices and edges are called its *elements*. Let $G = (V, X)$ be a graph. For a vertex v and an edge x, v *m-dominates* x if $x \in \langle N[v] \rangle$ and x $m-dominates$ v if $v \in N[x]$. A set $S \subseteq V$ is a $vertex-edge$ *dominating set* ($VED-set$) if every edge in G is m-dominated by a vertex in S. A set $F \subseteq X$ is an $edge-vertex$ *dominating set* ($EVD-set$) if every vertex in G is m-dominated by an edge in F. The $vertex-edge$ *domination number* $\gamma_{ve} = \gamma_{ve}(G)$ and $edge-vertex$ *domination number* $\gamma_{ev} = \gamma_{ev}(G)$ of a graph G are respectively the cardinality of a minimum VED-set and EVD-set of G. The minimal number of cliques that cover all the vertices of a graph is well known in graph theory as partition number $\theta(G)$ introduced by Berge [1]. It was also called as vertex clique covering number by Choudam et al [2]. They studied its edge analogue, edge clique covering number $\theta_1(G)$ defined as the minimum number of cliques that cover all the edges of a graph G.

A vertex $v \in V$ block dominates(b-dominates) a block $b \in B(G)$ if v is incident on b. A set $D \subseteq V$ is said to be a *vertex block dominating set* (VBD) if every block in G is b-dominated by some vertex in D. The *vertex block domination number* $\gamma_{vb} = \gamma_{vb}(G)$ is the cardinality of a minimum vertex block dominating set of G. Similarly a set $F \subseteq B(G)$ is said to be a *block vertex dominating set* (BVD) if every vertex in G is b-dominated by some block in F. The *block vertex domination number* $\gamma_{bv} = \gamma_{bv}(G)$ is the cardinality of a minimum block vertex dominating set of G. Two blocks $b_1, b_2 \in B(G)$ are said to $bb-dominate$ each other if there is a common cutvertex incident with b_1 and b_2. A set $L \subseteq B(G)$is said to be a *bb-dominating set*(BBD-set) if every block in G is bb-dominated by some block in L. The *bb-domination number* $\gamma_{bb} = \gamma_{bb}(G)$ is the cardinality of a minimum BBD set of G. Two blocks are said to be *bb-independent* if they have no vertex in common. A set $L \subseteq B(G)$ is said to be *bb-independent set*(BBI-set) if no two blocks in L have a vertex in common. The *bb-independence number* $\beta_{bb} = \beta_{bb}(G)$ is the maximum number of blocks in a BBI set of G.

1.1. Block path, Block star and Block complete graphs. A B-path is a sequence of blocks and cutvertices say, $b_1, c_2, b_3, c_4, b_5, ..., b_{m-2}, c_{m-1}, b_m$ beginning and ending with blocks in which each cutvertex c_i is distinct and incident with the blocks b_{i-1}, b_{i+1}. The length of a B-path is the number of cutvertices in it. Any B-path with m blocks is denoted as BP_m and has $m-1$ cutvertices. Note that we cannot define a B-cycle for, if the starting and ending blocks of a B-path are same then the entire graph will reduce to a single block. A graph G with m blocks is B-complete graph B_m if any two blocks of G are adjacent. A graph G is a B-star denoted by $B_{m_1,m_2,...,m_k}$, if there exists a block b with k cutvertices and i^{th} cutvertex is incident with $m_i + 1$ blocks, $m_i \in N, 1 \leq i \leq k$. A graph G is a block-tree (B-tree) if there exists a unique B-path between any two blocks of a graph. Since block-cutvertex graph of any graph is a tree we observe that *every connected graph G with m blocks and n cutvertices is a B- tree.* The edges of a B-tree are the blocks of G with multiple (two or more) vertices. A single block is a B-tree with no cutvertex. Thus a B-Tree is a generalized version of a tree with vertices of a B-tree as cutvertices of G and edges as the blocks of G. Vareities of B-trees such as B-path with 5 blocks BP_5, B-complete graph B_5, B-star $B_{1,2,1,3,1}$ are shown in Figure 1 Removal of a block b from G is equivalent to removing all

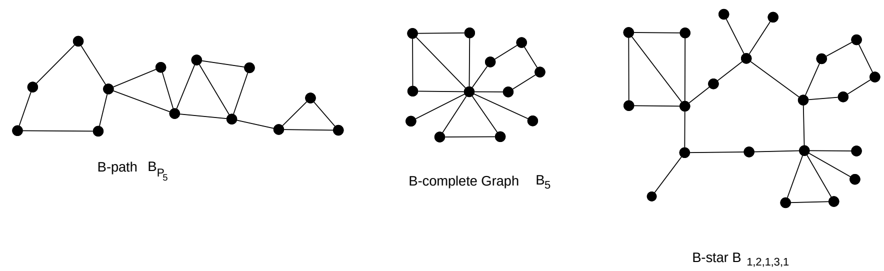

Figure 1. Varieties of Block Trees

the noncutvertices and edges incident on the block b. Note that a path P_5 is a unique graph with 5 vertices. But the block BP_5 is not unique in the sense that in the example above the block c_5 can be replaced by any other block and we still call it as BP_5. Infact, BP_5 reperesents a class of B-paths with 5 blocks. Similarly B-complete and B-stars are not unique.

By a Hypergraph H as defined by Berge [3] we mean an ordered pair, (X, ϕ) where X is a nonempty finite set whose elements are called vertices of H and ϕ is a set of subsets of X called edges of H . Thus a graph is a hypergraph $H = (X, \phi)$ with $|E| = 2$ for every $E \in \phi$. On the otherhand a graph itself is a hypergraph which can be visualized by viewing the edges as the blocks of G. Hence if we consider $G = (X, \phi)$ with $|b| = n$ for every $b \in \phi$ where b is any block of G, then G behaves like a hypergraph (in the sense of Berge).

2. Inverse Block Domination in Graphs

Definition 2.1. *Let S be a minimum VBD set of G. A VBD set $S_1 \subseteq V - S$ is called an inverse vertex block dominating set of G. The* inverse domination number $\gamma_{vb}^{-1} = \gamma_{vb}^{-1}(G)$ *is the cardinality of a minimum inverse VBD set of G.*

Example 2.1.

Note that $\gamma_{vb}^{-1}(G)$ need not exists for every graph. If G admits an inverse VBD set, then G is called VB invertible graph. For example any double star, shown in Figure 2 does not have an inverse dominating set. The corona $K_3 \circ K_1$ is VB invertible as it admits an inverse VBD set. Infact any two pendent vertices together with a non adjacent nonpendant vertex forms a minimum inverse VBD

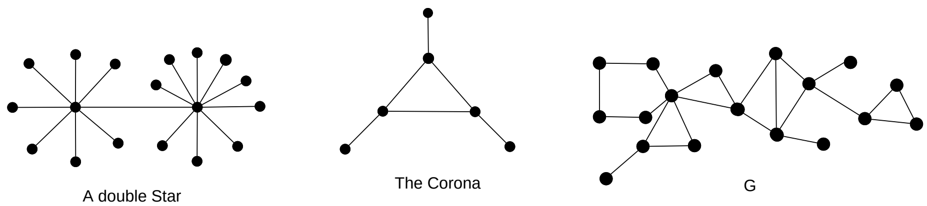

FIGURE 2

set. For the graph G, in Figure 2, $\gamma_{vb}(G) = 5$ and $\gamma_{vb}^{-1}(G) = 6$. Therefore G is VB invertible. Any path P_n is VB Invertible.

Proposition 2.2. *A graph G is VB invertible if and only if there exists a minimum VBD set S such that S is VB independent.*

Proof. Suppose G is VB invertible. Then there exists a inverse VBD set S_1 of G with respect to the minimum VBD set S.

Claim: S is VB independent. For, if S is not VB independent , then $\langle S \rangle$ has atleast one block of G. But then this block is not VB dominated by any vertices in S_1, a contradiction to the statement that S_1 is a VBD set of G. Hence our claim.

Conversely, suppose S be a VBD set of G which is VB independent. Then $V - S$ has at least one vertex from each block and hence $V - S$ is an inverse VBD set of G. Thus G is VB Invertible.

If G is a graph such that every block has atleast one noncutvertex, then the set of all noncut-vertices forms an inverse VBD set of G and therefore G is VB-invertible. □

3. Inverse VB Independent number

Definition 3.1. *et D be a maximum VB independent set of G. Then a VB independent set $D_1 \subseteq V - D$ is called an inverse VB independent set of G with respect to D. The inverse VB independent number $\beta_{vb}^{-1} = \beta_{vb}^{-1}(G)$ is the cardinality of a maximum inverse VB independent set of G.*

Note that $\beta_{vb}^{-1}(G)$ exists for every graph. Infact , inverse VB independent set of G is the maximum VB independent set of $\langle V - D \rangle$ where D is a maximum VB independent set of G.

Proposition 3.2. *For any block path BP_m with m blocks,*

$$\gamma_{vb}^{-1}(BP_m) = \left\lceil \frac{m+1}{2} \right\rceil$$

$$\beta_{vb}^{-1}(BP_m) = \left\lfloor \frac{m}{2} \right\rfloor$$

Proof. Let BP_m be a path with m blocks. Then the $m-1$ cut-vertices are labeled as $1, 2, 3, \ldots, m-1$. Let a be any noncutvertx in the first block and b be any noncutvertx in the last block of the path BP_m. For even m, let $D = \{1, 3, 5, \ldots, m-1\}$

is a minimum VBD set of G. Then $\{a, 2, 4, \ldots, m-2, b\}$ is a minimum inverse VBD set of G of cardinality $\left\lceil \frac{m+1}{2} \right\rceil$. On the other hand for odd m, Let $D = \{1, 3, 5, \ldots, m-2, b\}$ is a minimum VBD set of G. Then $\{a, 2, 4, \ldots, m-1\}$ is a minimum inverse VBD set of G of cardinality $\left\lceil \frac{m+1}{2} \right\rceil$. Hence the result follows in the first case. The second result follows from the fact that, for any block path BP_m, the maximum inverse VB independent set is the set $D_1 = \{1, 3, 5, \ldots, m-1\}$ or $D_2 = \{1, 3, 5, \ldots, m-2, b\}$ according as m is even or odd of cardinality $\left\lfloor \frac{m}{2} \right\rfloor$. □

Proposition 3.3. *For any graph G, the following inequality chain holds.*

$$\beta_{vb}^{-1}(G) \leq \gamma_{vb}(G) \leq \gamma_{vb}^{-1}(G) \leq \beta_{vb}(G)$$

Proof. Since any maximum inverse VB independent set is the subset of a minimum VBD set we have $\beta_{vb}^{-1}(G) \leq \gamma_{vb}(G)$ and as any inverse VBD set is a subset of maximum VB independent set we have $\gamma_{vb}^{-1}(G) \leq \beta_{vb}(G)$. Finally, as every inverse VBD set is also a VBD set we have $\gamma_{vb}(G) \leq \gamma_{vb}^{-1}(G)$. □

Proposition 3.4. *For any VB invertible graph G,*

$$\gamma_{vb}^{-1}(G) \leq \frac{p\Delta_{vb} - m}{\Delta_{vb}}$$

Proof. We first recall that $\frac{m}{\Delta_{vb}} \leq \gamma_{vb}(G)$. Further if S is a minimum VBD set of a VB invertible graph G, then $V - S$ is an inverse VBD set of G.

Therefore $\gamma_{vb}^{-1}(G) \leq |V - S| \leq p - \gamma_{vb}(G) \leq p - \frac{m}{\Delta_{vb}} = \frac{p\Delta_{vb} - m}{\Delta_{vb}}$. □

Proposition 3.5. *For any VB invertible graph G with m blocks, $\gamma_{vb}^{-1} \leq m$.*

Proof. Let v_i be a noncutvertx in the block b_i for $1 \leq i \leq m$. Then $S = \{v_1, v_2, \ldots, v_m\}$ is an inverse block dominating set of G. Therefore $\gamma_{vb}^{-1} \leq |S| = m$. Hence the proof. □

4. Inverse BBD Number and Inverse BBI Number.

Definition 4.1. *Let G be any graph and L be the minimum BBD set of G. A BBD set $L_1 \subseteq B - L$ is called an* inverse BBD set *of G with respect to L. The order of a minimum inverse dominating set of G is called the inverse BBD number of G denoted as $\gamma_{bb}^{-1} = \gamma_{bb}^{-1}(G)$.*

We observe that $\gamma_{bb}(G) \leq \gamma_{bb}^{-1}(G)$

Definition 4.2. *Let G be any graph and L be the maximum BBI set of G. A BBI set $L_1 \subseteq B - L$ is called an* inverse BBI set *of G with respect to L. The order of a maximum inverse BBI set of G is called the inverse BBI number of G denoted as $\beta_{bb}^{-1} = \beta_{bb}^{-1}(G)$.*

We observe that $\beta_{bb}(G) \geq \beta_{bb}^{-1}(G)$.

Example 4.1. *For the double star in Figure 2,* $\gamma_{bb}^{-1} = 2$ *and* $\beta_{bb}^{-1} = 2$.
Infact for any double star , $\gamma_{bb}^{-1}(K_{1,n} * K_{1,m}) = 2$ *and* $\beta_{bb}^{-1}(K_{1,n} * K_{1,m}) = 2$.
For the Corona, $\gamma_{bb}^{-1}(K_3 \circ K_1) = 3$ *and* $\beta_{bb}^{-1}(K_3 \circ K_1) = 1$.
In fact for any Corona $G_1 \circ G_2$, $\gamma_{bb}^{-1}(G_1 \circ G_2) = |V(G_2)|$ *and* $\beta_{bb}^{-1}(G_1 \circ G_2) = 1$.
For the graph G in Figure 2, $\gamma_{bb}^{-1}(G) = 4$ *and* $\beta_{bb}^{-1}(G) = 2$

Here we recall the Theorems which appear in [9]

Theorem 4.3. *For any graph G with maximum bb-degree* Δ_{bb}

$$\frac{m}{1 + \Delta_{bb}(G)} \leq \gamma_{bb}(G) \leq m - \Delta_{bb}(G)$$

Further the bound is sharp.

Proposition 4.4. *Let G be any graph with m blocks and maximum VB degree* Δ_{vb}*, then*

$$\left\lceil \frac{m}{\Delta_{vb}} \right\rceil \leq \gamma_{vb}$$

Further the bound is sharp.

Proposition 4.5. *For any connected graph G,*

$$\gamma_{vb}(G) = \beta_{bb}(G)$$

Proposition 4.6. *For any block path BPm with m blocks,*

$$\gamma_{bb}^{-1}(BP_m) = \left\lceil \frac{m+1}{3} \right\rceil$$

$$\beta_{bb}^{-1}(BP_m) = \left\lceil \frac{m}{3} \right\rceil$$

Proposition 4.7. *For any graph G with m blocks,*

$$\gamma_{bb}^{-1}(G) \leq \frac{m\Delta_{bb}}{\Delta_{bb} + 1}$$

Proof. If L is a minimal BBD set of G, then $B(G) - L$ is also a BBD set of G. Therefore any inverse BBD set $L_1 \subseteq B(G) - L$. From Theorem 4.3 we have, $\gamma_{bb}(G) \geq \dfrac{m}{\Delta_{bb} + 1}$.

Therefore $\gamma_{bb}^{-1}(G) = |L_1| \leq |B(G) - L| = m - \gamma_{bb}(G) \leq m - \dfrac{m}{\Delta_{bb} + 1} = \dfrac{m\Delta_{bb}}{\Delta_{bb} + 1}$. □

Proposition 4.8. *For any graph G with m blocks,*

$$\beta_{bb}^{-1}(G) \leq \frac{m(\Delta_{vb} - 1)}{\Delta_{vb}}$$

Proof. From Proposition 4.4 and Proposition 4.5 we know that $\dfrac{m}{\Delta_{vb}} \leq \gamma_{vb} = \beta_{bb}$. If L is any BBI set of G then any inverse BBI set $L_1 \subseteq B(G) - L$. Therefore $\beta_{bb}^{-1}(G) = |L_1| \leq |B(G) - L| = m - \beta_{bb}(G) \leq m - \dfrac{m}{\Delta_{vb}} = \dfrac{m(\Delta_{vb} - 1)}{\Delta_{vb}}$. □

References

[1] C. Berge, *Theory of Graphs and its Applications*, Methuen, London, 1962.

[2] S. A. Choudam, K. R. Parthasarathy and G. Ravindra, *Line-clique cover number of a graph*, Proceedings of INSA, **41(3)**,(1975), 289-293

[3] G.S.Domke, J.E Dunbar and Lisa S Markus, *Inverse Dominating sets, Ars Combinatoria*, **72**(2004) , 113-118.

[4] F. Harary, *Graph Theory*, Addison Wesley,1969.

[5] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Domination In Graphs: Advanced Topics*, Marcel Dekker, Inc., N.Y., 1997

[6] S.M.Hedetnimi, S.T Hedetnimi, Renu C.Laskar, P.J.Slater, *Disjoint dominating sets*, Proceedings of Int. conf. on Disc. Math. (ICDM 2006) 88-102.

[7] V.R.Kulli and S.C.Sigarakanti, *Inverse Domination in Graphs*, National Academy Science Letters, **14** (1991),473-475.

[8] E. Sampathkumar and S. S. Kamath, *Mixed Domination in Graphs*, Sankhya, (Special Volume) **54**, (1992), 399-402.

[9] Surekha and P. G. Bhat, *Mixed Block Domination in Graphs*, Journal of International Academy of Physical Sciences, **15**, (2011), 345-357.

[10] T. Tamizh Chelvam and G.S. Grace Prema, *Equality of Domination and Inverse Domination*, ARS Combinatoria, **95**(2010), 103-111.

[11] D. B. West, *Introduction to Graph Theory,* Prentice Hall, 1996.

Surekha Ravishankar Bhat:Department of Mathematics, G. Shankar Government First Grade Women's College, Ajjarkad, Udupi, 576101, India

Email address: surekharbhat@gmail.com

R. S. Bhat: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India.

Email address: ravishankar.bhats@gmail.com

Smitha Ganesh Bhat: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India.

Email address: smitha.holla@manipal.edu