Global and Stochastic Analysis
Vol. 11 No. 3 (June, 2024)

Received: 20th April 2024 *Revised: 30th May 2024* *Accepted: 7th June 2024*

PRIME PERFECT IDEALS IN SEMINEARRINGS

KEDUKODI BABUSHRI SRINIVAS, KAVITHA KOPPULA* AND KUNCHAM SYAM PRASAD

Abstract. In this paper, we define different prime perfect ideals of a right seminearring M and corresponding prime radicals. Then prove the relationship between prime ideals and are illustrated with the suitable examples. Further, we prove that, if $P_e(M)$ is the intersection of equiprime perfect ideals of M, then $P_e = \{M \mid P_e(M) = M\}$ is a Kurosh-Amitsur radical class. In addition, we prove results on c-prime perfect ideals and corresponding radicals.

1. Introduction

A seminearring is an algebraic system which forms a semigroup with respect to the binary operations multiplication ('·') and addition ('+') satisfies one of the (right or left) distributive law. Hoorn and Rootsellar [12] considered the kernel of a seminearring homomorphism is the ideal of a seminearring. By using this ideal definition, different types of prime ideals in seminearrings are defined by Javed [1, 2]. The concept of equiprime ideal of a nearing was defined by Booth, Gronewald and Veldsman [4]. Then they proved that in nearrings, if the equiprime radical is an intersection of all equiprime ideals, it will lead to a KA (Kurosh-Amitsur) radical class. Subsequently, Veldsman [13] explained the equiprimeness in nearrings and the relationship with various types of primeness in nearrings. Completely prime radical is defined and explained that in the class of all non-zero nearrings without divisors of zero it coincides with the upper radical by Groenewald [7]. The relationship between different types of prime ideals and prime radicals in nearrings was discussed by Birkenmeier, Heatherly and Lee [3].
In the present paper, we define various prime perfect ideals and radicals in seminearrings. Koppula, Kedukodi and Kuncham [9] defined strong ideal of a seminearring. Later, prime strong ideals, corresponding radicals in seminearrings were defined and related results were obtained. Further, Koppula, Kedukodi and Kuncham [8] defined the concept of perfect ideal of a seminearring and proved the standard isomorphism theorems. The two concepts perfect and strong ideal of a seminearring coincides in nearrings and rings. In this paper, we provide an example of perfect ideal of a seminearring, which is not a strong ideal. Then we define various prime perfect ideals of seminearrings and prime radicals in seminearrings.

2020 *Mathematics Subject Classification.* 16Y30; 16Y60; 20M122.
Key words and phrases. seminearring, perfect ideal, prime ideal, radical.
* Corresponding Author.

Further, we proved results related to KA radical class. In addition, we obtain results on c-prime ideals and corresponding radicals.

2. Preliminaries

In the present section, we provide basic definitions and results which are useful to obtain the present manuscript results.

Definition 2.1. [12] An algebraic structure $(M, +, \cdot)$ is said to be a right seminearring, if the below mentioned conditions are satisfied.

(1) M is a semigroup with respect to addition.
(2) M is a semigroup with respect to multiplication.
(3) $(m_1 + m_2)m_3 = m_1m_3 + m_2m_3$ for all $m_1, m_2, m_3 \in M$.
(4) $m + 0 = 0 + m = m$ for all $m \in M$.
(5) $0m = 0$ for all $m \in M$.

In the present paper, all seminearrings are considered as right seminearrings and M denotes a right seminearring.

The following definition is actually defined for semirings (Golan[6]) and now, it is adopted for seminearrings.

Definition 2.2. Let E be any non-empty subset of M. For $m_1, m_2 \in M$,
$m_1 \ _E\!\equiv m_2$ implies there exist $a_1, a_2 \in E$ such that $m_1 + a_1 = m_2 + a_2$.

Definition 2.3. [8] Let $\phi \neq E$ be a subset of M. Then E is a perfect ideal of M, if the below mentioned conditions hold.

(1) For $e_1, e_2 \in E,\ e_1 + e_2 \in E$.
(2) For $m \in M,\ \ E + m \subseteq m + E$.
(3) If $m_1 \ _E\!\equiv m_2$ then $m_1 \in m_2 + E$, where $m_1, m_2 \in M$.
(4) $Em \subseteq E$ for all $m \in M$.
(5) $m(m' + E) \subseteq mm' + E$ for all $m, m' \in M$.

In condition 3 of the Definition 2.3,
If $m_2 = 0$ then we have, if $m_1 \ _I\!\equiv 0$ then $m_1 \in E$.
If $m_1 = 0$ and $m_2 = 0$ then we have, if $0 \ _E\!\equiv 0$ then $0 \in E$.
In the following, seminearring homomorphism means seminearring perfect homomorphism [8].

Proposition 2.4. [8] If E is a perfect ideal of M then $\psi : M \to M/E$ is an onto seminearring homomorphism.
If $\pi : M \to R$ is an onto seminearring homomorphism, then $ker\ \pi$ is a perfect ideal of M.

Theorem 2.5. [8] If E and F are perfect ideals of M then $E \cap F$ is a perfect ideal of F and $(E + F)/E \cong F/(E \cap F)$.

Theorem 2.6. [8] If E and F are perfect ideals of M and $E \subseteq F$ then
$M/F \cong (M/E)/(F/E)$.

Definitions 2.7, 2.8, 2.9, 2.10 and 2.11 are redefined for seminearrings from the original nearring definitions (Pilz[11]).

Definition 2.7. The constant part of a seminearring M is
$M_c = \{m \in M \mid mm' = m, \text{ for all } m' \in M\}$.

Definition 2.8. The zero-symmetric part of a seminearring M is
$M_0 = \{m \in M \mid m0 = 0\}$.

Definition 2.9. Let E be any nonempty subset of a semigroup $(M, +)$. Then E is said to be a subsemigroup of M, if $m_1, m_2 \in E$ then $m_1 + m_2 \in E$.

Definition 2.10. Let M be a seminearring and E be an additive subsemigroup of M. Then E said to be a subseminearring of M, if $0 \in E$ and $EE \subseteq E$.

Definition 2.11. Let M be a seminearring and E be a subseminearring of M. Then E is said to be left invariant (right invariant, respectively) if $EM \subseteq E$ ($ME \subseteq E$, respectively). If E is both right and left invariant, then E is said to be invariant.

The following Definitions are actually defined for rings and are taken from the Gardner and Weigandt[5]. Now, we redefined them for seminearrings.

In the following, η denotes the class of seminearrings.

Definition 2.12. [5] The class η is said to be an hereditary, if $M \in \eta$ and E is an ideal of M then $E \in \eta$.

Definition 2.13. [5] A class of seminearrings η is said to be regular, if $0 \neq E$ is an ideal of M and $M \in \eta$ implies that $E \twoheadrightarrow F \neq 0$ (E has a non-zero homomorphic image F) such that $F \in \eta$.

From the above two definitions, it is clear that hereditary implies regularity.

Definition 2.14. [5] Let E be an ideal of a seminearring M. Then E is said to be an essential ideal of M if there exists an ideal $0 \neq F$ of M such that $E \cap F \neq 0$ and is denoted by $E \triangleleft \cdot M$.

The following definition is taken from Booth[4] and now, we redefined it for seminearrings.

Definition 2.15. [9] Let M be a seminearring. Then M is said to be an equiprime if

(1) $\forall\ 0 \neq m_1, m_2 \in M,\ m_1 M m_2 \neq (0)$.
(2) If $(0) \neq T$ is any invariant subsemigroup of M and $m_1, m_2 \in M$ then $tm_1 = tm_2, \forall\ \ t \in T$ implies $m_1 = m_2$.

3. Prime perfect ideals

In this section, we define prime perfect ideals of seminearring and identified the relationship among the prime perfect ideals. These ideals are explained with the suitable examples.

Remark 3.1. From the Definition 2.3, the following conditions hold.

(1) $E + m \subseteq m + E \Leftrightarrow m + E = E + m + E$
(2) $m(m' + E) + E = mm' + E \Leftrightarrow m(m' + E) \subseteq mm' + E$ for all $m, m' \in M$.

Proof. 1. First, we assume that $E + m \subseteq m + E$, for all $m \in M$.
Let $m \in M$ be arbitrarily fixed. Now, take $z \in E + m + E$.
Then $z = e_1 + m + e_2$, for some $e_1, e_2 \in E$. As $E + m \subseteq m + E$, there exists $e_3 \in E$ such that $z = (e_1 + m) + e_2 = (m + e_3) + e_2 \in m + E$. Clearly, we have $m + E \subseteq E + m + E$. Hence $m + E = E + m + E$.
Now, we assume that $m + E = E + m + E$. Let $y \in E + m$.
Then $y = e + m$, for some $e \in E$. By using the given condition, there exists $e_5 \in E$ such that $y = e + m + 0 = m + e_5 \in m + E$.
2. First, we assume that $m(m' + E) \subseteq mm' + E$, for all $m, m' \in M$.
Let $m, m' \in M$ be arbitrarily fixed. Now, take $y \in m(m' + E) + E$.
This implies there exist $a_1, a_2 \in E$ such that $y = m(m' + a_1) + a_2$.
By using the given condition, there exists $a_3 \in E$ such that
$y = m(m' + a_1) + a_2 = (mm' + a_3) + a_2 \in mm' + E$.
Clearly, we get $mm' + E \subseteq m(m' + E) + E$.
Now, take $z \in m(m' + E)$. Then there exists $e \in E$ such that
$z = m(m' + e) + 0 = mm' + e' \in mm' + E$ for some $e' \in E$. □

Definition 3.2. A perfect ideal E of M is said to be a c-prime perfect if $m_1, m_2 \in M$ with $m_1 m_2 \in E$ then either $m_1 \in E$ or $m_2 \in E$.

Definition 3.3. A perfect ideal E of M is said to be a completely semiprime perfect ideal if $m \in M$ with $m^n \in E$ (n is a positive integer) then $m \in E$.

Definition 3.4. A perfect ideal E of M is said to be an equiprime perfect if $s, a, b \in M$ with $sma \,{}_E\!\equiv smb \ \forall\, m \in M$ then either $s \in E$ or $a \,{}_E\!\equiv b$.

Definition 3.5. A perfect ideal E of M is said to be a 3-prime perfect if $s, t \in M$ with $smt \in E \ \forall\, m \in M$ then either $s \in E$ or $t \in E$.

Example 3.6. Let $M = \{0, m_1, m_2, m_3\}$ be a set with respect to $+$ and $\cdot$ are defined as mentioned in the following tables.

$+$	0	m_1	m_2	m_3
0	0	m_1	m_2	m_3
m_1	m_1	0	m_2	m_3
m_2	m_2	m_3	m_2	m_3
m_3	m_3	m_2	m_2	m_3

.	0	m_1	m_2	m_3
0	0	0	0	0
m_1	0	0	m_1	m_1
m_2	0	0	m_2	m_2
m_3	0	0	m_3	m_3

Then M is a seminearring. Now, take $E = \{0, m_1\}$. Then E is a perfect ideal of M.
Now, $m_2 + \{0, m_1\} = \{m_2, m_3\}$.
$\{0, m_1\} + m_2 = m_2$. Therefore $m_2 + \{0, m_1\} \nsubseteq \{0, m_1\} + m_2$. Hence E is not a strong ideal of M.
The perfect ideal E partitions M into the equivalence classes as
$0/E = m_1/E = \{0, m_1\}, \ m_2/E = m_3/E = \{m_2, m_3\}$.
In addition, E is a c-prime, 3-prime and an equiprime perfect ideal of M.

Proposition 3.7. If E is an equiprime perfect ideal of M then $M_c \subseteq E$.

Proof. Let $a \in M_c$. Then $aa' = a \;\; \forall\, a' \in M$. Now, take $b \in M$.
Then $aa'b + 0 = aa'0 + 0 \;\forall\, a' \in M$. This implies $aa'b \;{}_E\!\equiv aa'0 \;\forall\, a' \;\in M$.
As E is an equiprime perfect ideal, we get $a \in E$ or $b \;{}_E\!\equiv 0$.
This gives $a \in E$ or $b \in E$. If E is a proper perfect ideal then $b \in E$ is a contradiction. Therefore $a \in E$. Thus $M_c \subseteq E$. ☐

Theorem 3.8. If E is an equiprime perfect ideal of M then E is a 3-prime perfect ideal.

Proof. Let $x, y \in M$ be such that $xmy \in E$ for all $m \in M$.
If $x \in E$, then E is a 3-prime perfect.
Suppose $x \notin E$. As $M_c \subseteq E$, we have $xm0 \in E$ for all $m \in M$.
Now, fix $m \in M$. Then $xmy \in xm0 + E$. This gives $xmy \;{}_E\!\equiv xm0$.
As $m \in M$ is arbitrary, we have $xmy \;{}_E\!\equiv xm0 \;\forall\, m \in M$. Because E is an equiprime perfect and $x \notin E$, we get $y \equiv_E 0 \Rightarrow y \in E$.
Thus E is a 3-prime . ☐

Here we provide an example to show that every 3-prime perfect ideal need not be equiprime perfect.

Example 3.9. Let $M = \{0, m_1, m_2, m_3\}$ be a set with respect to $+$ and $\cdot$ defined as follows:

$+$	0	m_1	m_2	m_3
0	0	m_1	m_2	m_3
m_1	m_1	m_1	m_1	m_3
m_2	m_2	m_1	0	m_3
m_3	m_3	m_3	m_3	m_3

.	0	m_1	m_2	m_3
0	0	0	0	0
m_1	m_1	m_1	m_1	m_1
m_2	m_2	m_2	m_2	m_2
m_3	m_3	m_3	m_3	m_3

Then M is a seminearring and $E = \{0, m_2\}$ is a 3-prime, c-prime perfect ideal of M. However, E is not an equiprime perfect ideal of M. Because $m_1 m m_1 {}_E\!\equiv m_1 m m_2 \;\forall m \in M$, but $a \notin E$ and $m_1 {}_E\!\not\equiv m_2$.

Proposition 3.10. If E and F are perfect ideals of M such that $x + (E \cap F) = (x + E) \cap (x + F)$ for all $x \in M$ then $E \cap F$ is a perfect ideal of M.

In the following sections, we consider that, if E and F are perfect ideals of M then $x + (E \cap F) = (x + E) \cap (x + F) \;\forall\, x \in M$.

4. Kurosh-Amitsur prime radical

Here we redefined the Kurosh-Amitsur radical class for seminearrings and it is actually taken from [5].
In the following, E is considered as a perfect ideal of a seminearring M.

Definition 4.1. η is said to be a KA radical class if the below mentioned conditions are satisfied.

(1) η is homomorphically closed.
(2) $\eta(M) = \Sigma(E \triangleleft M \mid E \in \eta)$ is in η, for every seminearring M.
(3) For every seminearring M, $\eta(M/\eta(M)) = 0$.

Proposition 4.2. If the class η satisfies conditions 1. and 2. of the Definition 4.1, then 3. is equivalent to:
$\overline{3}$. If E is a perfect ideal of the seminearring M and E, $M/E \in \eta$ then $M \in \eta$ (closed under extensions).

Proof. Suppose that the class η satisfies condition 3.
Now, take seminearrings E, M/E from η. Then by 2, we have $E \subseteq \eta(M)$. By using Isomorphism theorem, we get $\frac{M/E}{\eta(M)/E} \cong M/\eta(M)$. As $M/E \in \eta$, by condition 1. we get $\frac{M/E}{\eta(M)/E} \in \eta$. Again by condition 1., we have $M/\eta(M) \in \eta$.
Then $M/\eta(M) = \eta(M/\eta(M)) = 0$ (by 3). This implies $M = \eta(M) \in \eta$. Therefore $M \in \eta$. For the converse, we assume that $\overline{3}$. holds and $\eta(M/\eta(M)) \neq 0$.
Then there exists a perfect ideal $F/\eta(M) \in \eta$ of $M/\eta(M)$ such that $F/\eta(M) \neq 0$.
As $\eta(M)$ and $F/\eta(M)$ are from η, then by $\overline{3}$. we get $F \in \eta$.
Therefore by 2., we have $F \subseteq \eta(M)$. Then $F = \eta(M)$. This implies $F/\eta(M) = 0$.
As this is a contradiction for $\eta(M/\eta(M)) \neq 0$, hence we get $\eta(M/\eta(M)) = 0$. □

Proposition 4.3. If η satisfies the conditions 1.(Definition 4.1) and $\overline{3}$. then the condition 2. (Definition 4.1) is equivalent to $\overline{2}$., which is defined as follows.
$\overline{2}$. If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_\alpha \subseteq \cdots$ is an ascending chain of perfect ideals of a seminearring S, if each E_α is in η then $\cup E_\alpha$ is also in η (η has the inductive property).

Proof. Suppose that the condition 2. holds. Now, take $G = \cup E_\alpha$.
As each $E_\alpha \in \eta$, we have each $E_\alpha \subseteq \eta(G) = \Sigma\{E_\alpha \triangleleft G \mid E_\alpha \in \eta\}$.
Hence $G = \cup E_\alpha \subseteq \eta(G)$. Then by condition 2. we get $G = \eta(G)$ is in η.
Suppose that $\overline{2}$. holds. Then by using Zorn's lemma, there exists a maximal η-ideal G of M. Now, take H is any other η-ideal of M.
Then, we have $(G+H)/H \cong G/(G \cap H)$.
Because $G \to G/(G \cap H)$ is an onto homomorphism, $G \in \eta$, then by condition 1., we have $G/(G \cap H)$ is in η. Again by condition 1., we have $(G+H)/H$ is in η.
Therefore by condition $\overline{3}$, $G + H \in \eta$. Because G is maximal perfect ideal, we get $G + H$ is G. This implies $\eta(M) \subseteq G$. Thus $\eta(M) = G$ is in η. □

Theorem 4.4. The class η is a radical class iff the below conditions hold.

(1) η is homomorphically closed
$\overline{2}$. If $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_\alpha \subseteq \cdots$ is an ascending chain of perfect ideals of a seminearring S, if each E_α is in η, then $\cup E_\alpha$ is also in η (η has inductive property).
$\overline{3}$. If E is a perfect ideal of the seminearring M and E, $M/E \in \eta$ then $M \in \eta$.

(η is closed under extensions).

Theorem 4.5. If η is a class of seminearrings, then the following statements are equivalent.

(I). η is a radical class.
(II). (A) If M is in η then for $M \twoheadrightarrow G \neq 0$ there exists a perfect ideal H of G such that $0 \neq H \in \eta$.

(B) For $M \multimap G \neq 0$ there is a perfect ideal H of G such that $0 \neq H \in \eta$ then $M \in \eta$.

(III). η satisfies II(A), has the inductive property and is closed under extensions.

Proof. First we show that (I) implies (III). Let M be in η.
Then for any $M \to G \neq 0$ onto homomorphism, we have $0 \neq G \in \eta$.
This implies $0 \neq G = \eta(G) \in \eta$. Therefore η satisfies II(A) and by Theorem 4.4, η has the inductive property and is closed under extensions.
Now, we will show that (III) implies (II). It is sufficient to prove that η satisfies II(B). Suppose that M is in η and for any onto homomorphism $M \to G \neq 0$, H is a perfect ideal of G such that $0 \neq H \in \eta$, then M is not in η.
By Zorn's lemma there exists a maximal ideal E of M, as η has inductive property, we get $E \in \eta$. This implies $M/E \neq 0$. Because $M/E \neq 0$ is a homomorphic image of M, then by II(A) there exists an ideal F/E of M/E such that $0 \neq F/E \in \eta$.
As $E \in \eta$ and $F/E \in \eta$, by inductive property we get $F \in \eta$. Which is a contradiction for E is a maximal ideal. Hence $M \in \eta$.
Now, we show that (II) implies (I). Let $M \in \eta$ and G be a nonzero homomorphic image of M. Then we show that $G \in \eta$. Now, take H is a homomorphic image of G. Then by II(A), there exists an ideal V of H such that $0 \neq V \in \eta$.
Now, by II(B), we get $G \in \eta$. Therefore η is homomorphically closed.
Now, take $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_\alpha \subseteq \cdots$ is an ascending chain of ideals of η, each $E_\alpha \in \eta$. Then we prove that $\cup E_\alpha$ is in η.
Let $\cup E_\alpha/F$ be a nonzero seminearring. Then there exists an index α such that $E_\alpha \nsubseteq F$. This implies $0 \neq (E_\alpha + F)/F$ and we know that $(E_\alpha + F)/F$ is an ideal of $\cup E_\alpha/F$. As $E_\alpha \in \eta$, the homomorphic image of E_α is $E_\alpha/E_\alpha \cap F$ in η.
As $\frac{E_\alpha}{E_\alpha \cap F} \cong \frac{E_\alpha + F}{F}$, we get $\frac{E_\alpha + F}{F} \in \eta$. Now, we have $\cup E_\alpha \to \cup E_\alpha/F \neq 0$ is an onto homomorphism and the ideal of $\cup E_\alpha/F$ is $0 \neq (E_\alpha + F)/F \in \eta$.
Then by II(B), we get $\cup E_\alpha$ is in η. Thus η has the inductive property.
Now, take E and M/E are in η. Then we show that $M \in \eta$.
Let M/F be a nonzero seminearring.
Case(i): If $E \subseteq F$. Then $\frac{M/E}{F/E}$ is a homomorphic image of M/E. As $M/E \in \eta$, we get $\frac{M/E}{F/E} \in \eta$. Then by Isomorphism theorem we get, $M/F \in \eta$.
Hence by II(A) there exists an ideal $0 \neq K/F$ of M/F such that $K/F \in \eta$.
Case(ii): If $E \nsubseteq F$. Then $0 \neq (E + F)/F$ is an ideal of M/F. As E is in η, we have $E/E \cap F$ is in η and we know that $E/(E \cap F) \cong (E + F)/F$. Hence $(E + F)/F \in \eta$.
In two cases $0 \neq M/F$ has a non-zero ideal in η. Hence by II(B), we get $M \in \eta$.
Therefore η is closed under extensions. □

Theorem 4.6. If η is regular, then $U\eta = \{M \mid M \multimap N \neq 0 \text{ such that } N \notin \eta\}$ is a radical class.

In the following, the class of $c-$prime seminearrings is denoted by η_c, the class of equiprime seminearrings by η_e and the class of 3-prime seminearrings by η_3'.

Definition 4.7. The equiprime radical is $P_e'(M) = \cap\{E \triangleleft M \mid M/E \in \eta_e\}$, 3-prime radical is $P_3'(M) = \cap\{E \triangleleft M \mid M/E \in \eta_3'\}$ and c-prime radical is $P_c'(M) = \cap\{E \triangleleft M \mid M/E \in \eta_c\}$.

Proposition 4.8. [9] The class of equiprime seminearrings η_e is hereditary on invariant subsemigroups. Particularly, the class η_e is hereditary.

Definition 4.9. Let E be a perfect ideal of F, F is a left invariant perfect ideal of M and $F/E \in \eta$, $m_1, m_2 \in M$. Then η is said to satisfy U_1',

(1) If $(m_1m)/E = (m_2m)/E,\ \forall\ m \in F$, then $m_1 \in m_2 + E$.
(2) If $(mm_1)/E = (mm_2)/E,\ \forall\ m \in F$, then $m_1 \in m_2 + E$.

Proposition 4.10. If F is a perfect ideal of M and left invariant such that $F/E \in \eta$ and η satisfies the condition U_1', then E is a perfect ideal of M.

Proof. Clearly, $x + y \in E, \forall\ x, y \in E$. Now, take $y \in E + m_1 + E$.
Then there exist $e_1, e_2 \in E$ such that $y = e_1 + m_1 + e_2$.
Let $m \in F$. Then $ym = (e_1 + m_1 + e_2)m = e_1m + m_1m + e_2m$. As E is a perfect ideal of F, there exists $e_3 \in E$ such that $e_1m + m_1m + e_2m = m_1m + e_3$.
That is, $ym/E = m_1m/E$. Then by Definition 4.9(1), we get $y \in m_1 + E$.
This implies $E + m_1 + E \subseteq m_1 + E$. As $0 \in E$, we have $m_1 + E \subseteq E + m_1 + E$.
Hence we get $E + m_1 + E = m_1 + E$.
Let $x\ {}_E{\equiv}\ y$. Then there exist $e_1, e_2 \in E$ such that $x + e_1 = y + e_2$.
Let $m \in F$. Then $(x + e_1)m = (y + e_2)m$
$\Rightarrow xm + e_1m = ym + e_2m.$
$\Rightarrow xm + e_3 = ym + e_4\ [e_1m = e_3 \in E, e_2m = e_4 \in E].$
$\Rightarrow xm/E = ym/E.$
As m is arbitrary, we have $xm/E = ym/E,\ \forall m \in F$. Then by Definition 4.9(1), we get $x \in y + E$. Now, we will prove that $Em_1 \subseteq E, \forall\ m_1 \in M$.
Let $y \in Em_1$. Then $y = am_1$, for some $a \in E$. Now, take $m \in F$.
Then $(am_1)m = a(m_1m) \in aF \subseteq E \Rightarrow (am_1)m = e_2 + 0m$, for some $e_2 \in E$.
This gives $(am_1)m/E = om/E$. Then by Definition 4.9(1), we get $am_1 \in E$.
Now, we show that $m_1(s' + E) + E = m_1s' + E,\ \forall\ m_1, s' \in M$.
Let $a, e_1 \in E$ and $m \in F$. Suppose we assume that $(m_1(s' + a))m/E \neq m_1s'm/E$.
Now, take $x \in F$. Then $x((m_1(s' + a))m) = x(m_1(s'm + am)) = xss'm + e_2$, for some $e_2 \in E$. This implies $x((m_1(s' + a))m)/E = x(m_1s'm)/E$.
As this is a contradiction for $(m_1(s'+a))m/E \neq m_1s'm/E$, hence we get $(m_1(s'+a))m/E = (m_1s'm)/E,\ \forall\ m \in F$.
Then by Definition 4.9(1), we get $m_1(s' + a) \in m_1s' + E$.
This implies $m_1(s' + E) + E = m_1s' + E$. Thus E is a perfect ideal of M.
□

Proposition 4.11. If F is a perfect ideal of M and left invariant such that $F/E \in \eta$ and the class η satisfies the condition U_1', then $(E : F)_M$ is a perfect ideal of M.

Proof. By Proposition 4.10, we have E is a perfect ideal of M.
Now, take $m_1, m_2 \in (E : F)_M$. Then $m_1F \subseteq E$ and $m_2F \subseteq E$.
This implies $m_1F + m_2F \subseteq E + E = E$. That is, $(m_1 + m_2)F \subseteq E$.
Hence $m_1 + m_2 \in (E : F)_M$.
Let $z \in (E : F)_M + x + (E : F)_M$. Then there exist $m_1, m_2 \in (E : F)_M$ such that $z = m_1 + x + m_2$. Now, take $m \in F$.
Then $zm = (m_1+x+m_2)m = m_1m+xm+m_2m$. As E is a perfect ideal of F, there

exists $m_3 \in E$ such that $m_1m + xm + m_2m = xm + m_3$. That is, $zm/E = xm/E$.
As $m \in F$ is arbitrary, we have $zm/E = xm/E, \ \forall \ m \in F$.
Then by Definition 4.9(1), we get $z \in x + E \subseteq x + (E : F)_M$.
Hence $(E : F)_M + x + (E : F)_M = x + (E : F)_M$.
Let $z \in x(x' + (E : F)_M) + (E : F)_M$. Then there exist $m_1, m_2 \in (E : F)_M$ such that $z = x(x' + m_1) + m_2$. Now, take $m \in F$.
Then $zm = (x(x' + m_1) + m_2)m = x(x' + m_1)m + m_2m = x(x'm + m_1m) + m_2m$.
As E is an ideal of M, there exist $a_1 \in E$ such that $x(x'm + m_1m) + m_2m = xx'm + a_1$. This implies $zm/E = xx'm/E$.
Then by Definition 4.9(1), we get $z \in xx' + E \subseteq xx' + (E : F)_M$.
Hence $x(x' + (E : F)_M) + (E : F)_M = xx' + (E : F)_M$.
Let $x \equiv_{(E:F)_M} y$. Then there exist $y_1, y_2 \in (E : F)_M$ such that $x + y_1 = y + y_2$.
Now, take $m \in F$. Then $(x + y_1)m = (y + y_2)m$.
$\Rightarrow xm + y_1m = ym + y_2m \Rightarrow xm/E = ym/E$.
Then by Definition 4.9(1), we get $x \in E \subseteq (E : F)_M$.
Now, we will prove that $(E : F)_M M \subseteq (E : F)_M$. Let $z \in (E : F)_M M$.
Then there exists $y \in (E : F)_M$ and $m \in M$ such that $z = ym$.
Then $zF = (ym)F = y(mF) \subseteq yF \subseteq E$. Hence $z \in (E : F)_M$.
Thus $(E : F)_M$ is a perfect ideal of M. □

Proposition 4.12. If the class η_e satisfies the condition U_1' and E is a left invariant perfect ideal of the seminearring M and $E/F \in \eta_e$, then $(F : E)_M$ is an equiprime perfect ideal of M.

Proof. By Proposition 4.11, we have $(F : E)_M$ is a perfect ideal of M.
Now, take $m_1, m_2 \in M$ such that $m_1, m_2 \notin (F : E)_M$.
Then $m_1a \notin F$ and $m_2b \notin F$, for some $a, b \in E$.
As F is an equiprime perfect ideal of E, we have $m_1aEm_2b \nsubseteq F$.
Hence $m_1Ey \nsubseteq (F : E)_M$.
Now, take T is any invariant subsemigroup of $M \ni (F : E)_M \subset T$.
Now, take $m_1, m_2 \in M$ such that $am_1/(F : E)_M = am_2/(F : E)_M \ \ \forall \ a \in T$.
Then there exist $f_1, f_2 \in (F : E)_M$ such that $am_1 + f_1 = am_2 + f_2$
$\Rightarrow (am_1 + f_1)d = (am_2 + f_2)d \ \forall \ d \in E$
$\Rightarrow am_1d + f_1d = am_2d + +f_2d$
$\Rightarrow am_1d/F = am_2d/F$.
As $F \subseteq (F : E)_M \subset T$, F is an equiprime perfect ideal of E and $a(m_1d)/F = a(m_2d)/F$, we get $m_1d/F = m_2d/F \ \forall \ d \in E$.
Then by Definition 4.9(1), we get $m_1 \in m_2 + F \subseteq y + (F : E)_M$.
Hence $m_1/(F : E)_M = m_2/(F : E)_M$.
Then by the Definition 2.15, we have $(F : E)_M$ is an equiprime perfect ideal of M. □

Lemma 4.13. If the class η_e satisfies the condition U_1' and $E \lhd \cdot M$ such that $ME \subseteq E$, $E \in \eta_e$ and $0 \neq a \in S$, then $aE \neq 0$ and $Ea \neq 0$.

By Proposition 4.8, we have η_e is hereditary. Then by Proposition 4.6, we get $U\eta_e$ is a KA radical class.

Theorem 4.14. $U\eta_e = P_e' = \{S \mid S \text{ is a seminearring such that } P_e'(S) = S\}$.

Theorem 4.15. $U\eta_c = P'_c = \{M \mid M \text{ is a seminearring such that } P_c(M) = M\}$.

Proof. Let $M \in U\eta_c$. Then $M \multimap N \neq 0$ such that $N \notin \eta_c$. This means M has no non-zero completely prime ideals. This implies $M = P'_c(M)$. Therefore $U\eta_c \subseteq P'_c$. Now, take $M \in P'_c$. Then $M = P'_c(M))$. This implies M has no nonzero c-prime perfect ideals of M.
Hence $M \in U\eta_c$. Thus we get $U\eta_c = P_c$. □

Proposition 4.16. If E is an essential ideal of a seminearring M, M is zero-symmetric and the class η_c satisfies condition U'_1, $E \in \eta_c$ then $M \in \eta_c$ (η_c is closed under essential extensions).

Proof. From Proposition 4.11, we have $(0 : E)_M = \{s \in M \mid sE = 0\}$ is a perfect ideal of M. Now, $((0 : E)_M \cap E)^2 \subseteq (0 : E)_M E = (0)$.
As $(0 : E)_M \cap E$ is an ideal of E and $E \in \eta_c$, we get $(0 : E)_M \cap E = (0)$.
Because E is an essential ideal of M, then we get $(0 : E)_M = (0)$.
Now, take $m_1, m_2 \in M$ such that $m_1, m_2 \neq 0$.
As $(0 : E)_M = (0)$, there exists $x, y \in E$ such that $m_1 x \neq 0$ and $m_2 y \neq 0$.
Suppose that $xm_1 = 0$. Then $m_1 x m_1 x = m_1 0 x = 0$. As $E \in \eta_c$, we get $m_1 x = 0$.
Which is a contradiction to the assumption $xm_1 = 0$.
Therefore $xm_1 \neq 0$. Then we get $(xm_1)(m_2 y) \neq 0$.
This gives $m_1 m_2 \neq 0$. Thus $M \in \eta_c$. □

Proposition 4.17. If E is a completely semiprime perfect ideal of M and left invariant, $x, y \in M$, then the following conditions hold.

(1) If $xy \in E$ then $yx \in E$.
(2) If $xy \in E$ and $a \in M$ then $xay \in E$.

Proof. 1. Let $(yx)^2 \in E$. Then $(yx)^2 = (yx)(yx) = y(xy)x$. As $xy \in E$, there exists $i_1 \in E$ such that $y(xy)x = y(i_1 x) = yi_2 \in E$. As E is completely semiprime, we get $yx \in E$.
2. Now, $(xay)^2 = (xay)(xay) = xa(yx)ay = xai_1ay$ (From 1.), for some $i_1 \in E$. As E is left invariant, we get $(xay)^2 \in E$. Because E is completely semiprime, we get $xay \in E$. □

Proposition 4.18. If M is an equiprime seminearring then $mx = my$, $\forall\, m \in M$ implies $x = y$.

Proof. Suppose that $mx = my$, $\forall\, m \in M$. This implies $xmx = xmy$, $\forall\, m \in M$. As M is an equiprime seminearring, we get $x = 0$ or $x = y$.
If $x \neq y$ then $xmx = xmy$, $\forall\, m \in M$ implies $x = 0$ and $ymx = ymy$, $\forall\, m \in M$ implies $y = 0$, which is a contradiction. Therefore $x = y$. □

Acknowledgment. All authors acknowledge Manipal Institute of Technology, Manipal Academy of Higher Education, Karnataka, India for the encouragement and support.

References

1. Ahsan, J.: Seminear-rings characterized by their S-ideals. I, *Proc. Japan Acad. Ser. A* **71** (1995) 101–103.
2. Ahsan, J.: Seminear-rings characterized by their S-ideals. II, *Proc. Japan Acad. Ser. A* **71** (1995) 111–113.
3. Birkenmeier, G., Heatherly, H., Lee, E.: Completely prime ideals and radicals in nearrings in Near-Rings and Near-Fields, Springer publishers, 1995.
4. Booth, G.L., Groenewald, N.J., Veldsman, S.: A Kurosh-Amitsur prime radical for near-rings, *Comm. Algebra.* **18** (1990) 3111–3122.
5. Gardner, B.J., Wiegandt, R.: *Radical theory of rings*, CRC Press, 2003.
6. Golan, J.S.: *Semirings and their Applications*, Kluwer Acadamic Publishers, Dordrecht (1999)
7. Groenewald, N.J.: The completely prime radical in near-rings, *Acta Math. Hung.* **51** (1988) 301–305.
8. Koppula, K., Kedukodi, B.S., Kuncham, S.P.: On perfect ideals of seminearrings, *Beitr Algebra Geom* **62** (2021) 823–842.
9. Koppula, K., Kedukodi, B.S., Kuncham, S.P.: On prime strong ideals of a seminearring, *Mat. Vesn.* **72** (3) (2020) 243–256.
10. Koppula, K., Kedukodi, B.S., Kuncham, S.P.: Congruences in seminearrings and their correspondence with strong ideals, accepted for publication in *Algebr. Struct.*
11. Pilz, G.: *Near-Rings: The theory and its applications*, Revised edition, North Hollond, 1983.
12. Van Hoorn, W.G., Van Rootselaar, B.: Fundamental notions in the theory of seminearrings, *Compos. Math.*, **18** (1967) 65–78.
13. Veldsman, S.: On equiprime near-rings, *Comm. Algebra.* **20** (1992) 2569–2587.

Kedukodi Babushri Srinivas: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.
Email address: `babushrisrinivas.k@manipal.edu`

Kavitha Koppula: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.
Email address: `kavitha.koppula@manipal.edu`

Kuncham Syam Prasad: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.
Email address: `syamprasad.k@manipal.edu`