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ON DIFFERENT ENERGIES OF PRINCIPAL IDEAL GRAPHS OF RECTANGULAR BANDS

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Abstract. For a semigroup $\mathbb{S}$, the principal left(right) ideal graph ${}_S\Gamma$ (Γ_S) is a graph with vertex set $\mathbb{S}$ and two elements are adjacent if and only if their principal left (right) ideals intersects. In this paper we determined different types of graph energies such as A-energy, Q-energy, and L-energy of the principal left (right) ideal graphs of rectangular bands and established the relation between these energies. The automorphism groups of principal left (right) ideal graphs of such semigroups are also determined.

1. Introduction

Let $\mathbb{S}$ be a semigroup and $\alpha \in \mathbb{S}$. The principal left ideal generated by α, denoted by $\mathbb{S}^1\alpha$ is defined by $\mathbb{S}^1\alpha = \{s\alpha : s \in \mathbb{S}\} \bigcup \{\alpha\}$. The principal right ideal is defined accordingly. The principal left(right) ideal graph ${}_S\Gamma$ (Γ_S)of a semigroup $\mathbb{S}$ is defined as a graph with vertex set $\mathbb{S}$ in which two distinct vertices are adjacent if and only if their principal left (right) ideals intersects[7]. In [9] and [**?**] Indu and John characterised the principal ideal graphs of the rectangular bands and Rees matrix semigroups respectively. A semigroup $\mathbb{S}$ satisfying $\alpha^2 = \alpha$ for all $\alpha \in \mathbb{S}$ is said to be a band. If $\alpha\beta\alpha = \alpha$ for all $\alpha, \beta \in \mathbb{S}$ then the band $\mathbb{S}$ is said to a rectangular band. For any two nonvoid sets $\mathcal{I}$ and T, define an operation on $\mathcal{I} \times T$ by

$$(i, t)(j, s) = (i, s),$$

then $\mathcal{I} \times T$ is a rectangular band and any rectangular band is isomorphic to $\mathcal{I} \times T$ for some $\mathcal{I}$ and T [6]. We refer the reader to [6] for un-cited terms and concepts regarding semigroups and [2] for graphs. Throughout this paper $\mathbb{S} = \mathcal{I} \times T$ denotes a rectangular band.

In this work (Section 2) we describe A-eigenvalues [11], L-eigenvalues [1, 10], and Q-eigenvalues [12] of ${}_S\Gamma$, the eigenvalues of adjacency marix $A({}_S\Gamma)$, Laplacian matrix $L({}_S\Gamma)$ and signless Laplacian matrix $Q({}_S\Gamma)$ respectively. In Section 3 we describe and compare different energies such as A-energy $\varepsilon_A({}_S\Gamma)$, the energy of the adjacencny matrix [5, 11], L-energy $\varepsilon_L({}_S\Gamma)$, the energy of the Laplacian matrix [13], and Q-energy $\varepsilon_Q({}_S\Gamma)$, the energy of the signless Laplacian matrix [3]. An automorphism [4] of a graph Γ is a bijection $\psi : V(\Gamma) \to V(\Gamma)$ such that

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ψ preserves both adjacency and non-adjacency. In Section 4 the automorphism group of ${}_S\Gamma$ and Γ_S are characterised in the case of rectangular bands.

Since two principal left ideals of $\mathbb{S} = \mathcal{I} \times T$ intersects if and only if they are equal, we have the following lemmas by Indu and John [9] which are used the sequel.

Lemma 1.1. [9] *Let $\mathbb{S} = \mathcal{I} \times T$ be a rectangular band. Then the principal left ideal graph ${}_S\Gamma$ is a disconnected graph with $o(T)$ components in which each connected block is a complete graph with $o(\mathcal{I})$ vertices.*

Lemma 1.2. [9] *Let $S = \mathcal{I} \times T$ be a rectangular band. Then the principal right ideal graph Γ_S is a disconnected graph with $o(\mathcal{I})$ components in which each connected is a complete graph with $o(T)$ vertices.*

2. Spectrum of ${}_S\Gamma$ and Γ_S

In this section we sketch out the characteristic polynomials of different matrices derived from the principal ideal graphs ${}_S\Gamma$ and Γ_S of rectangular band $\mathcal{I} \times T$. This will help to identify different energies of ${}_S\Gamma$ and Γ_S.

First we charactrise the A-energy of the principal ideal graph.

Theorem 2.1. *Consider $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$, and $o(T) = n$. Then the A-energy $\varepsilon_A({}_S\Gamma)$ of the principal left ideal graph is $2n(m-1)$.*

Proof. By Lemma 1.1 ${}_S\Gamma$ is a disconnected graph with $o(T) = n$ components in which each component is complete with $o(\mathcal{I}) = m$ vertices. So the adjacency matrix $A({}_S\Gamma)$ is given by

$$A({}_S\Gamma) = \begin{bmatrix} J_m - I_m & \mathcal{O}_m & \mathcal{O}_m & \cdots & \mathcal{O}_m \\ \mathcal{O}_m & J_m - I_m & \mathcal{O}_m & \cdots & \mathcal{O}_m \\ \vdots & \vdots & \vdots & & \vdots \\ \mathcal{O}_m & \mathcal{O}_m & \mathcal{O}_m & \cdots & J_m - I_m \end{bmatrix}_{mn \times mn}$$

where J_m, I_m and $\mathcal{O}_m$denotes the all-ones matrix, identity matrix, and the zero matrix of order m respectively. Each $J_m - I_m$ block is given by

$$J_m - I_m = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}_{m \times m}$$

The characteristic polynomial of each nonzero block $J_m - I_m$ is

$$[x - (m-1)](x+1)^{(m-1)}$$

Since there are $o(T)$ identical blocks, the characteristic polynomial of $A({}_S\Gamma)$ is

$$[x - (m-1)]^n (x+1)^{n(m-1)}$$

Hence the A-eigenvalues of $A({}_S\Gamma)$ are $(m-1)$ of multiplicity n and (-1) of multiplicity $n(m-1)$ and the A-energy $\varepsilon_A({}_S\Gamma) = 2n(m-1)$. □

In a similar manner using Lemma 1.2 and Theorem 2.1 we can characterise the A-energy of the principal right ideal graph as follows.

Corollary 2.2. *Consider $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$, and $o(T) = n$. Then the A-energy $\varepsilon_A(\Gamma_S)$ of the principal right ideal graph is $2m(n-1)$.*

Next we give a characterisation for the largest A-eigenvalue of ${}_S\Gamma$,

Theorem 2.3. *Let $\rho({}_S\Gamma)$ be the largest A-eigenvalue of ${}_S\Gamma$, where $\mathbb{S} = \mathcal{I} \times T$. Then we have the following.*

(i) $\rho({}_S\Gamma) \geq 0$ *and* $\rho({}_S\Gamma) = o(\mathcal{I}) - 1$.
(ii) *In ${}_S\Gamma$, the multiplicity of $\rho({}_S\Gamma)$ is $o(T)$.*

Proof. By Theorem 2.1, $\rho({}_S\Gamma) = o(\mathcal{I}) - 1$. Since $\mathcal{I}$ is nonempty, $o(\mathcal{I}) \geq 1$. Hence $\rho({}_S\Gamma) \geq 0$ and this proves (i). Proof of (ii) follows from Theorem 2.1. □

Now we state the dual case of Theorem 2.3, the proof is similar so omitted.

Corollary 2.4. *Let $\rho(\Gamma_S)$ be the largest A-eigenvalue of Γ_S, where $\mathbb{S} = \mathcal{I} \times T$. Then,*

(i) $\rho(\Gamma_S) \geq 0$ *and* $\rho(\Gamma_S) = o(T) - 1$.
(ii) In Γ_S, the multiplicity of $\rho(\Gamma_S)$ is $o(\mathcal{I})$. □

Now we have the description of L-energy for ${}_S\Gamma$.

Theorem 2.5. *Consdier $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$ and $o(T) = n$. Then the L-energy, $\varepsilon_L({}_S\Gamma)$, of the principal left ideal graph is $mn(m-1)$.*

Proof. By Lemma 1.1, the Laplacian matrix $L({}_S\Gamma)$ is a square matrix given by

$$L({}_S\Gamma) = \begin{bmatrix} mI_m - J_m & \mathcal{O}_m & \mathcal{O}_m & \dots & \mathcal{O}_m \\ \mathcal{O}_m & mI_m - J_m & \mathcal{O}_m & \dots & \mathcal{O}_{gm} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathcal{O}_m & \mathcal{O}_m & \mathcal{O}_m & \dots & mI_m - J_m \end{bmatrix}_{mn \times mn}$$

where each diagonal block is

$$mI_m - J_m = \begin{bmatrix} m-1 & -1 & -1 & \cdots & -1 \\ -1 & m-1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & -1 & -1 & \dots & m-1 \end{bmatrix}_{m \times m}$$

The characteristic polynomial of each nonzero block $mI_m - J_m$ is $x(x-m)^{m-1}$. Since there are $o(T)$ identical blocks, the characteristic equation of $L({}_S\Gamma)$ is

$$x^n(x-m)^{n(m-1)} = 0.$$

Hence L-eigenvalues of $L({}_S\Gamma)$ are m and 0 of multiplicity $n(m-1)$ and n respectively; and the L-energy $\varepsilon_L({}_S\Gamma) = mn(m-1)$. □

Similarly using Lemma 1.2 and Theorem 2.5, we have the following result.

Corollary 2.6. *Consider $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$ and $o(T) = n$. Then the L-energy $\varepsilon_L(\Gamma_S)$ of the principal right ideal graph is $mn(n-1)$.*

In the following result, we give a characterisation for the largest L-eigenvalue of ${}_S\Gamma$

Theorem 2.7. *Let $\mu({}_S\Gamma)$ be the largest L-eigenvalue of ${}_S\Gamma$, where $\mathbb{S} = \mathcal{I} \times T$. Then the following statements hold:*

(i) *$\mu({}_S\Gamma) \geq 1$ and $\mu({}_S\Gamma) = o(\mathcal{I})$.*
(ii) *In ${}_S\Gamma$, the multiplicity of the L-eigenvalue 0 is $o(T)$.*

Proof. By Theorem 2.5, $\mu({}_S\Gamma) = o(\mathcal{I})$. Since $\mathcal{I}$ is nonempty ,we have $o(\mathcal{I}) \geq 1$ and hence $\mu({}_S\Gamma) \geq 1$. This proves (i) and the second part follows from Theorem 2.5. □

Now we state the dual case of Theorem 2.7 without proof.

Corollary 2.8. *Let $\mu(\Gamma_S))$ be the largest L-eigenvalue of Γ_S, where $\mathbb{S} = \mathcal{I} \times T$. Then the following statements hold:*

(i) *$\mu(\Gamma_S) \geq 1$, and $\mu(\Gamma s) = o(T)$.*
(ii) *In Γ_S, the multiplicity of the L-eigenvalue 0 is $o(\mathcal{I})$*

Next Theroem describes the the Q-energy of principal left ideal graph.

Theorem 2.9. *Consider $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$ and $o(T) = n$. Then the Q-energy $\varepsilon_Q({}_S\Gamma)$ of the principal left ideal graph is $2n(m-1)$.*

Proof. By Lemma 1.1, the signless Laplacian matrix $Q({}_S\Gamma)$ is a square matrix given by

$$Q(s\Gamma) = \begin{bmatrix} (m-2)I_m + J_m & \mathcal{O}_m & \cdots & \mathcal{O}_m \\ \mathcal{O}_m & (m-2)I_m + J_m & \cdots & \mathcal{O}_m \\ \vdots & \vdots & & \vdots \\ \mathcal{O}_m & \mathcal{O}_m & \cdots & (m-2)I_m + J_m \end{bmatrix}_{mn \times mn}$$

where each diagonal block is

$$(m-2)I_m + J_m = \begin{bmatrix} m-1 & 1 & 1 & \cdots & 1 \\ 1 & m-1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \cdots & m-1 \end{bmatrix}_{m \times m}$$

The characteristic polynomial of each nonzero diagonal block is

$$x^{m-1}[x - (2m-2)]$$

. Since there are $o(T)$ identical blocks, the characteristic equation of $Q({}_S\Gamma)$ is

$$x^{n(m-1)}[x - (2m-2)]^n = 0$$

. Hence Q-eigenvalues of $Q({}_S\Gamma)$ are 0 of multiplicity $n(m-1)$ and $2(m-1)$ of multiplicity n and the Q-energy $\varepsilon_Q({}_S\Gamma) = 2n(m-1)$. □

The following Corollary is an immediate consequence of Lemma 1.2 and Theorem 2.9.

Corollary 2.10. *Consider $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = m$ and $o(T) = n$. Then the Q-energy $\varepsilon_Q(\Gamma_S)$ of the principal right ideal graph is $2m(n-1)$.*

The next result follows from Theorem 2.9.

Theorem 2.11. *Let $q({}_S\Gamma)$ be the largest Q-eigenvalue of ${}_S\Gamma$, where $\mathbb{S} = \mathcal{I} \times T$. Then we have the following:*

(i) *$q({}_S\Gamma) \geq 0$ and $q({}_S\Gamma) = 2(o(\mathcal{I}) - 1)$.*
(ii) *In ${}_S\Gamma$, the multiplicity of $q({}_S\Gamma)$ is $o(T)$.*

Dually we have the following Corollary.

Corollary 2.12. *Let $q(\Gamma_S)$ be the largest Q-eigenvalue of Γ_S, where $\mathbb{S} = \mathcal{I} \times T$. Then we have:*

(i) *$q(\Gamma_S) \geq 0$ and $q(\Gamma_S) = 2[o(T) - 1]$.*
(ii) *In Γ_S, the multiplicity of $q(\Gamma_S)$ is equal to $o(\mathcal{I})$.* □

3. Relationship between $\varepsilon_A({}_S\Gamma)$, $\varepsilon_L({}_S\Gamma)$, and $\varepsilon_Q({}_S\Gamma)$

In this section we establish the relation between A-energy, Q-energy, and L-energy of the principal left (right) ideal graphs of rectangular bands.

As a consequence of Theorem 2.1, and Theorem 2.9, we have the following result.

Theorem 3.1.

$$\varepsilon_A({}_S\Gamma) = \varepsilon_Q({}_S\Gamma)$$

By Corollary 2.2 and Corollary 2.10, we have following charactrisation.

Corollary 3.2.

$$\varepsilon_A(\Gamma_S) = \varepsilon_Q(\Gamma_S)$$

Theorem 3.3.

$$\varepsilon_L({}_S\Gamma) = \frac{o(\mathcal{I})}{2}\varepsilon_Q({}_S\Gamma)$$

Proof. From Theorem 2.5,
$\varepsilon_L({}_S\Gamma) = mn(m-1)$ and from Theorem 2.9, $\varepsilon_Q({}_S\Gamma) = 2n(m-1)$.
Hence $\varepsilon_L({}_S\Gamma) = \frac{m}{2}\varepsilon_Q({}_S\Gamma) = \frac{o(\mathcal{I})}{2}\varepsilon_Q({}_S\Gamma)$. □

Corollary 3.4.

$$\varepsilon_L(\Gamma_S) = \frac{o(T)}{2}\varepsilon_Q(\Gamma_S).$$

Proof. The result is obvious by Corollary 2.6 and Corollary 2.10. □

Theorem 3.5. *If $o(\mathcal{I}) = 2$, then $\varepsilon_L({}_S\Gamma) = \varepsilon_Q({}_S\Gamma)$.*

Proof. It is evident from Theorem 3.3. □

The following Corollary is an immediate consequence of 3.4.

Corollary 3.6. *If $o(T) = 2$, then $\varepsilon_L(\Gamma_S) = \varepsilon_Q(\Gamma_S)$*

3.1. Different energies of left(right) zero semigroups. A semigroup $\mathbb{S}$ is said to be a left (right) zero semigorup if $\alpha\beta = \alpha$ $(\alpha\beta = \beta)$ for all $\alpha, \beta \in \mathbb{S}$. These semigroups are special cases of rectangular bands, $\mathcal{I} \times T$, such that $o(T) = 1$ $(o(\mathcal{I}) = 1)$. So by substituting either $o(\mathcal{I}) = 1$ or $o(T) = 1$ in the results of the previous sections, we can deduce the corresponding results for left or right regular semigroups.

Theorem 3.7. *If $\mathbb{S}$ is a right zero semigroup, then the A-energy, L-energy, and Q-energy of ${}_S\Gamma$ are zeros.*

Proof. When $\mathbb{S}$ is a right zero semigroup, we can write $\mathbb{S} = \mathcal{I} \times T$ with $o(\mathcal{I}) = 1$. Now the result is clear from Theorem 2.1, Theorem 2.5, and Theorem 2.9 by substituting $o(\mathcal{I}) = 1$. □

As above we have similar result for Γ_S of left zero semigroups. We state it as a corollary without proof.

Corollary 3.8. *If $\mathbb{S}$ is left zero semigroup, then then A-energy, L-energy and Q-energy of Γ_S vanishes.* □

When $\mathbb{S}$ is a left zero semigroup, the pricipal right ideal graph Γ_S is a graph having $o(\mathbb{S})$ vertices and no edges. Also the principal left ideal graph ${}_S\Gamma$ is a complete graph with $o(\mathbb{S})$ vertices. Similarly when $\mathbb{S}$ is a right zero semigroup, Γ_S is a complete graph and ${}_S\Gamma$ is a null graph having $o(\mathbb{S})$ vertices.

4. Automorphism group of ${}_S\Gamma$

In this section we analyse the automorphism group, in the case of principal left(right) ideal graphs. Since ${}_S\Gamma$ is a disjoint union of complete graphs and each complete graphs having the same number of vertices, we get the follwoing result.

Theorem 4.1. *Let $\mathbb{S} = \mathcal{I} \times T$ be a rectangular band, then the automorphism group of ${}_S\Gamma$ is isomorphic to $S_n \times \prod_{k=1}^{n} S_m$, where S_m denotes the symmetric group on m letters and $o(\mathcal{I}) = m$, $o(T) = n$.*

Proof. Since ${}_S\Gamma$ is a disjoint union of K_m, complete graph with m vertices, we have the automorphism group of each component is isomorphic to S_m, Since there are $o(T)$ such components and we can also permute these components, we get the automorphism group isomorphic to

$$S_n \times \prod_{k=1}^{n} S_m.$$

□

Finally we state the dual case of Theorem 4.1.

Theorem 4.2. *Let $\mathbb{S} = \mathcal{I} \times T$ be a rectangular band , then the automorphism group of Γ_S is isomorphic to $S_m \times \prod_{k=1}^{m} S_n$.* □

Conclusion

The A-energy and Q-energy of principal left (right) ideal graphs of rectangular bands are same. Furthermore, the L-energy of ${}_S\Gamma$ is $\frac{o(\mathcal{I})}{2}$ times that of A-energy of ${}_S\Gamma$ and the L-energy of Γ_S is $\frac{o(T)}{2}$ times that of A-energy of Γ_S.

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