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A NOTE ON COMPLEMENTS IN A LATTICE

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Abstract. The concept of modular lattices naturally exhibit the links between the theoretical aspects of discrete structures and corresponding applications. In this paper, we consider the complements in a modular lattice with finite Goldie dimension (in short, FGD). We prove several properties and characterizations that involve θ-complement, θ-closed and relative θ-complemented, weak θ-complemented, etc in a lattice. We provide the necessary illustrations to justify the notions and generalizations in this paper.

1. Introduction

The notion of 'essential submodule' of a module over a ring is an analogy to the concept of 'dense subspace' in a topological space [2]. A submodule L is essential in a module M in case $K \cap L \neq (0)$, for each non-zero submodule K of M. However, as we know, a lattice need not contain a zero element, and so essentiality concept in a lattice with respect to an arbitrary element was introduced in [19]. Nevertheless, the concept of module over a ring is well interpreted in terms of the lattice structure of its submodules. Grzeszczuk and Puczylowski [9] established the idea of Goldie dimension from the module theory, to the modular lattices. They defined an essential element in a lattice with the least element 0. The theory has become significant and later many developments found in Calugareanu [6] wherein several ideas from modules over rings were generalized to the lattice theory. Goldie [10] introduced the concept of the Goldie dimension of modules over rings, and proved a characterization for a module to have finite Goldie dimension. Bhavanari [3] obtained several equivalent conditions in terms of descending chain conditions on essential submodules. There are good connections between semiprime ideals and uniform ideals of module over rings. Tapatee et.al [20, 21] studied relative essential ideals and relative complements and in [23], the authors studied the partial order aspects of modules over generalized rings. We refer to [19, 22] for the developments in modular lattices. The notion complement plays an important role in modules, specifically, as in [3, 21], to establish the dimension of a quotient submodule and the dimension of sum of two submodules. Analogously, in a lattice with 0, the notion pseudo-complement has been defined in [6], and some recent developments can be seen in [7]. Saki and Kiani [18] studied the properties of complements and pseudo-complements of finite modular lattices of subracks, and obtained some

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equivalent conditions. Rao and Beyene [16] have explored on irreducible elements in almost semilattices. The semi-complement undirected graph of lattice modules have been studied by Phadatare et.al [15].
In this paper, we consider a lattice $(L, \wedge, \vee)$ with the smallest element 0 and whenever necessary we assume 1 to be the greatest element in L. For $x, y \in L$ and $x \leq y$, the interval between x and y is denoted by $[x, y] = \{a \in L \mid x \leq a \leq y\}$, is a sublattice of L. If $a \neq 1$ in L, then a is called proper. In a bounded lattice an element a is called an atom (respectively, dual atom) if there is no $x \in L$ such that $0<x<a$ (respectively, $a<x<1$).
In this paper, we deal with the modular lattices and define θ-complement and weak θ-complement which generalize both the notions pseudo-complement and complement in L. We prove several properties as generalizations of results in [6, 13], wherein the lattice is upper continuous.

For comprehensive literature in lattice theory, we refer to [8].

The following definitions are from [1, 6].

A subset $\mathcal{D}$ of a poset is called upper directed, if each finite subset of $\mathcal{D}$ has an upper bound in $\mathcal{D}$. A complete lattice L is called upper continuous if $a \wedge (\bigvee \mathcal{D}) = \bigvee_{d \in \mathcal{D}} (a \wedge d)$ holds for every $a \in L$ and every upper directed subset $\mathcal{D} \subseteq L$. L is called modular if for any $x, y, z \in L$, $x \leq z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$. If $y \in L$ is maximal with respect to the property $x \wedge y = 0$, then y is called a pseudo-complement of x in L. L is pseudo-complemented if for every $x \in L$, there exists a pseudo-complement in L, and is relative pseudo-complemented, if each sublattice of L is pseudo-complemented. In a lattice with 0 and 1, $y \in L$ is called a complement of $x \in L$ if $x \wedge y = 0$ and $x \vee y = 1$.

2. θ-complement

Throughout, let $\theta \in L$ be an arbitrary but fixed element, where L is a modular lattice.

Definition 2.1. [19]

(1) $\theta \neq a \in L$ is θ-essential if $a \wedge x \neq \theta$ for every $\theta \neq x \in L$, we denote it as $a \leq_\theta^e L$. The set of all θ-essential elements in L is denoted by $E_\theta(L)$.
(2) Let $x \leq y \in L$. Then x is θ-essential in y if $x \leq_\theta^e [\theta, y]$. In other words, $x \wedge k \neq \theta$ for every $k \in (\theta, y]$, denoted by $x \leq_\theta^e y$. In this case, we call y as θ-essential extension of x.
(3) $a \in [x, y]$ is said to be θ-essential, if $\theta \in [x, y]$ and $a \wedge b \neq \theta$ for every $\theta \neq b \in [x, y]$.

Evidently, if $\theta = 0$, then the notion of 'θ-essential' coincides with the notion 'essential'.

Example 2.2. Let L be the lattice given in Fig. 1. Now $x \not\leq^e L$, for every $0 \neq x \in L$, whereas $x \leq_{\theta=b}^e L$, for every $\theta \neq x \in L$.

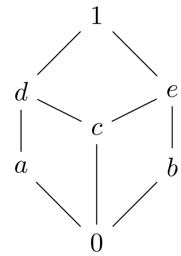

FIGURE 1

Example 2.3. [8] Consider the free distributive lattice L, on three generators given in Fig. 2. Then we have the following.

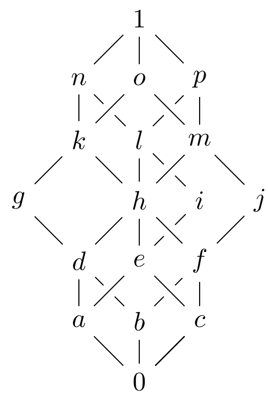

FIGURE 2

(1) $e \leq^e_{\theta=b} p$, whereas $e \nleq^e p$, since $e \wedge b = 0$ and $b \neq 0$.
(2) $i \leq^e_{\theta=b} p$, whereas $i \nleq^e p$, since $i \wedge b = 0$ and $b \neq 0$.

Definition 2.4. [6] A function $\phi : L_1 \to L_2$ between two lattices is called a lattice homomorphism if $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$, for all $x, y \in L_1$.

Theorem 2.5. [6] If $s, t \in L$, then $[t, (s \vee t)]$ and $[(s \wedge t), s]$ are isomorphic.

Lemma 2.6. Let $f : L_1 \to L_2$ be an isomorphism and $a \leq^e_\theta L_1$ implies $f(a) \leq^e_{f(\theta)} L_2$.

Proof. Let $a \leq^e_\theta L_1$. Let $b \in L_2$ such that $f(a) \wedge b = f(\theta)$. Then $a \wedge f^{-1}(b) = \theta$. Since $a \leq^e_\theta L_1$ and $f^{-1}(b) \in L$, we get $f^{-1}(b) = \theta$. Therefore, $b = f(\theta)$. This shows that $f(a) \leq^e_{f(\theta)} L_2$. □

Unlike in case of module over rings, essentiality need not be closed under homomorphic images. Indeed, in a lattice, the image of a θ-essential element under a

lattice homomorphism need not be θ-essential.
Consider the following example.

Example 2.7. Let L_1 and L_2 be two lattices given in Fig. 3. Let $f : L_1 \to L_2$ be a lattice homomorphism defined by $f(x) = x$, for $x \in \{0, a, b\}$ and $f(c) = 0$. Clearly, for $\theta = 0$, $a \leq^e_\theta L_1$, but $f(a) = a \not\leq^e_{f(\theta)=0} L_2$.

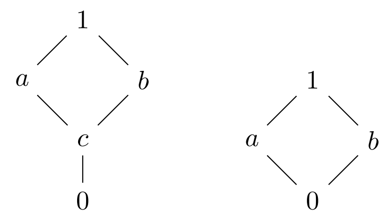

FIGURE 3. L_1 and L_2

Notation 2.8. If $a \leq^e_\theta c$, and $b \leq c$, then $a \leq^e_\theta b$.

Definition 2.9. [19] $S = \{a_i \mid i \in I,\ \text{where } I \text{ is finite}\} \subseteq L \setminus \{\theta\}$, is said to be θ-$\vee$-independent if $a_i \wedge \big(\bigvee_{j \neq i} a_j\big) = \theta$, for every $i \in I$.

Definition 2.10. For any $a, b \in L$, an element a is θ-closed in b, if a has no proper θ-essential extension in b, we denote it by $a \leq^{cl}_\theta b$.

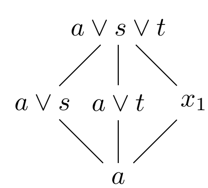

FIGURE 4

Proposition 2.11. For $\theta < a < b < c$ in L, if a is θ-closed in b and b is θ-closed in c, then a is θ-closed in c.

Proof. Suppose $a \leq^{cl}_\theta b$, $b \leq^{cl}_\theta c$, and $a \not\leq^{cl}_\theta c$. Then there exists $x \in L$ such that $a < x \leq^{cl}_\theta c$ and $a \leq^e_\theta x$. Now $x \wedge b \leq x$, implies $a \leq^e_\theta x \wedge b$. Since $a \leq^e_\theta x \wedge b \leq b$ and $a \leq^{cl}_\theta b$, we get $a = x \wedge b \ \cdots\ (1)$.
If $x \leq b$, then $a = x$, a contradiction. So, $x \not\leq b$, and so $b < b \vee x \leq c$. Since, $b \leq^{cl}_\theta c$, it follows that $b \not\leq^{cl}_\theta b \vee x$, thus there exists s such that $\theta < s \leq b \vee x$ and $s \wedge b = \theta$. Now, $a \leq^e_\theta x$, $s \wedge x \leq x$ and $a \wedge (s \wedge x) \leq a \wedge s < b \wedge s = \theta$, implies $s \wedge x = \theta \ \cdots\ (2)$.

If $a = (s \wedge x) \wedge b$, then

$$\begin{aligned} s &= s \wedge (b \vee x) \\ &= [s \wedge (s \wedge x)] \wedge (b \vee x) \\ &= s \wedge [((s \wedge x) \wedge b) \vee x], \text{ by modular law, } x \leq s \vee x \\ &= s \wedge (a \vee x) \\ &= s \wedge x \\ &= \theta, \end{aligned}$$

a contradiction. Hence, $a < (s \vee x) \wedge b \leq b$. Since $a \leq_\theta^{cl} b$, we get $a \not\leq_\theta^{cl} (s \vee x) \wedge b$, thus there exists t such that $\theta < t \leq (s \vee x) \wedge b$ and $a \wedge t = \theta \ \cdots$ (3).
Then from (1) it follows that $x \wedge t = x \wedge (b \wedge t) = a \wedge t = \theta$. Thus, if $s \wedge (x \wedge t) = \theta$, then the (s, t, x) is θ-$\vee$-independent, and thus $t \wedge (x \vee s) = \theta$, a contradiction, since $\theta < t \leq x \wedge s$. This shows that $s' = s \wedge (x \vee t) \neq \theta$.
Moreover, from (3) it follows,

$$\begin{aligned} x \vee s' &= x \vee (s \wedge (x \vee t)) \\ &= (x \vee s) \wedge (x \vee t), \text{ by modular law, } x \leq x \vee t \\ &= x \vee t \\ &\geq t \end{aligned}$$

So, we may replace s by s' without changing the validity of (3).
Therefore, we may assume that $\theta < s \leq x \vee t$ and $s \wedge b = \theta \ \cdots$ (4).
Now from (2) and (4), $x \vee s = x \vee t$. Also since $a \wedge t \leq b \leq b$ and $s \wedge b = \theta$, we have $s \wedge (a \wedge t) = \theta$. Since $a \wedge t = \theta$, and by modular law, $\{a, s, t\}$ is θ-$\vee$-independent. Moreover, since $a \leq x$ and by (2), $x \wedge (a \vee s) = a \vee (x \wedge s) = a \vee \theta = a$. Similarly, by using the equality $x \wedge t = \theta$, yields $x \wedge (a \vee t) = a$. Let $x_1 = x \wedge (a \vee s \vee t)$.
Next we claim that the elements $a \vee s$, $a \vee t$ and x_1 are the atoms of a lattice shown in Fig. 4, with bottom a and top $a \vee s \vee t$. Clearly, by modular law and the fact that $\{a, s, t\}$ is θ-$\vee$-independent, we get $(a \vee s) \wedge (a \vee t) = a \vee [s \wedge (a \vee t)] = a \vee \theta = a$. Again by modular law, since $a \leq x_1 \leq x$,

$$\begin{aligned} x_1 \wedge (a \vee s) &= a \vee (x_1 \wedge s) \\ &\leq a \vee (x \wedge s) \\ &= a \vee (x \wedge t) \\ &= x \wedge (a \vee t) \\ &= a. \end{aligned}$$

Now $x_1 \wedge (a \vee s) = a$, since $a \leq a \vee s$ and $a \leq x_1$. Clearly $x_1 \leq (a \vee s) \vee (a \vee t) = a \vee s \vee t$. Since $x \vee s = x \vee t \geq a \vee s \vee t$, we get $x_1 \vee (a \vee s) = (x \vee a \vee s) \wedge (a \vee s \vee t) = a \vee s \vee t$. Similarly, $x_1 \vee (a \vee t) = a \vee s \vee t$. From $a \wedge t = \theta$ and $t > \theta$, it follows that $a < a \vee t$, thus $a < x_1$. Now let $x_0 = x \wedge (s \vee t)$. Since $a \leq x$ and by modularity, we get

$$\begin{aligned} x_0 \vee a &= [x \wedge (s \vee t)] \vee a \\ &= x \wedge (a \vee s \vee t) \\ &= x_1 \\ &\geq a > \theta. \end{aligned}$$

Thus $x_0 \vee a > \theta$, hence $x_0 > \theta$.
Whereas,

$$\begin{aligned} a \wedge x_0 &= a \wedge x \wedge (s \vee t) \\ &= a \wedge (s \vee t), \text{ as } a \leq x \\ &= \theta, \end{aligned}$$

a contradiction to the assumption that $a \leq_\theta^e x$.

□

The converse of the Proposition 2.11 not necessarily true.

Example 2.12. Let L be the lattice given in Fig. 2. Then for $\theta = 0$, $d \leq_\theta^{cl} l$, $d \leq_\theta^{cl} h$, but $h \not\leq_\theta^{cl} l$, as $h \wedge x \neq 0$, for all $x \in [0, l]$.

The following definition is a generalization of pseudo-complement defined in [6].

Definition 2.13. $c \in L$ is called a θ-complement of b in L if c is maximal with respect to $b \wedge c = \theta$. Further, L is θ-complemented if every $x \in L$ has at least one θ-complement.

Example 2.14. Let $L = (D_{30}, \leq)$, the elements are positive divisors of 30, given in the Fig. 5. Write $x \leq y \Leftrightarrow x$ *divides* y, $x \vee y = \text{l.c.m}\{x, y\}$ and $x \wedge y = \text{g.c.d}\{x, y\}$. Then, d is a $(\theta = b)$-complement of f, but d is not a pseudo-complement of f,

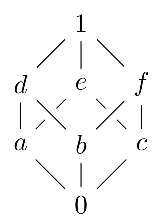

FIGURE 5

since $d \wedge f = b \neq 0$.

Definition 2.15. L is called relative θ-complemented if for every $x \in L$, $[\theta, x]$ is θ-complemented. Further, $x \in L$ is called a weak θ-complement if there exists $x' \in L$ such that $x \wedge x' = \theta$ and $x \vee x' = 1$. L is called weak θ-complemented if every $x \in L$ has at least one weak θ-complement in L.

Example 2.16. Let L be the non-modular lattice given in Fig. 6, of all subgroups of the group D_8, the dihedral group of order 8. Then, f is a $(\theta = c)$-complement of g, but f is not a pseudo-complement of g, since $f \wedge g = c \neq 0$.

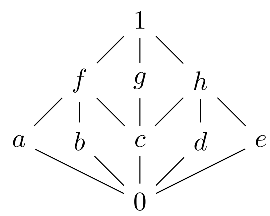

Figure 6

Example 2.17. Consider the Lattice L_1 given in Fig. 3. Here, for $(\theta = c)$, L_1 is θ-complemented, relative θ-complemented and weak θ-complemented. But L_1 is not pseudo-complemented or complemented, since $x \wedge y \neq 0$, for any $0 \neq x, y \in L_1$.

Lemma 2.18. Let $\theta \leq a \leq p \leq b$ be elements in L with 1, and r a weak θ-complement of p in L. Then $q = (a \vee r) \wedge b = a \vee (r \wedge b)$, a weak θ-complement of p in $[a, b]$.

Proof. Since r is weak θ-complement of p in L, we have $r \wedge p = \theta$ and $r \vee p = 1$. Now,

$$\begin{aligned} p \vee q &= p \vee a \vee (r \wedge b) \\ &= p \vee (r \wedge b) \\ &= (p \vee r) \wedge b, \text{ since } p \leq b \text{ and by modular law} \\ &= 1 \wedge b \\ &= b \end{aligned}$$

and

$$\begin{aligned} p \wedge q &= q \wedge p \\ &= (a \vee r) \wedge b \wedge p \\ &= (a \vee r) \wedge p \\ &= a \vee (r \wedge p), \text{ since } a \leq p \text{ and by modular law} \\ &= a \vee \theta \\ &= a. \end{aligned}$$

Therefore, q is a weak θ-complement of p in $[a, b]$. □

Corollary 2.19. A weak θ-complemented lattice is relative θ-complemented.

Proof. Follows from Lemma 2.18. □

Definition 2.20. L is called θ-inductive, if every sublattice $[x, y]$ of L satisfies the condition that: for any chain $\{b_i\}_{i \in I}$ in L and for any $a \in [x, y]$ with $a \wedge b_i = \theta$, for all $i \in I$, imply $a \wedge (\bigvee_{i \in I} b_i) = \theta$. If $\theta = 0$, θ-inductive coincides with the inductive defined in [6], and if L is upper continuous, then it is θ-inductive.

Notation 2.21. Every inductive lattice is θ-inductive. In a lattice of finite length, both inductive and θ-inductive exists. However, we may have an infinite lattice, which is θ-inductive but not inductive. Consider the infinite lattice $L = (\mathbb{Z}, \leq)$. Since the lattice has no least element, it is not inductive, whereas, L is $(\theta = 1)$-inductive.

For any θ-inductive lattice L and $\theta \leq a \in L$, the set $S = \{x \in L : a \wedge x = \theta,\ b \leq x\}$ has a maximal element by Zorn's lemma, which will be a θ-complement of a in L. Precisely, we state the following Lemma.

Lemma 2.22. Let L be θ-inductive lattice. Then every $\theta \leq a \in L$ has a θ-complement in L.

Corollary 2.23. If L is upper continuous, then every $\theta < a \in L$, has a θ-complement in L.

Proof. Follows from Lemma 2.22. □

Lemma 2.24. Let $1 \in L$, and $a, b \in L$. Then b is a θ-complement of a if and only if $a \wedge b = \theta$ and $a \vee b \leq_\theta^e [b, 1]$.

Proof. Let b be a θ-complement of a in L. Clearly, $b \wedge a = \theta$, and for any $d \in L$, $b < d$ implies $d \wedge a \neq \theta$. In particular, $d \in [b, 1]$, $d \neq b$ implies $d \neq \theta$. Then by modular law and since $d \wedge a \leq d \nleq b$, we have $\theta \leq b < b \vee (d \wedge a) = (a \vee b) \wedge d$. Hence $(a \vee b) \wedge d \neq \theta$, shows that $a \vee b \leq_\theta^e [b, 1]$. Conversely, suppose that $a \wedge b = \theta$ and $a \vee b \leq_\theta^e [b, 1]$. Then, for every $d \in [b, 1]$, $d \neq b$, we have $b < (a \vee b) \wedge d = b \vee (a \wedge d)$. That is, $a \wedge b = \theta$, and for every $b < d$, we have $a \wedge d \nleq b$. This implies $a \wedge d \neq \theta$. □

Corollary 2.25. Let $1 \in L$ and b be a θ-complement of $a \in L$, then $a \wedge b = \theta$ and $a \vee b \leq_\theta^e L$.

Proof. Let b be a θ-complement of a in L. Then by Lemma 2.24, we have $a \wedge b = \theta$ and $a \vee b \leq_\theta^e [b, 1]$. Then clearly, $\theta \leq b$. To show, $a \vee b \leq_\theta^e L$, take $d \in L$.
Case (i): If $\theta \neq d \leq b$, then $(a \vee b) \wedge d = d \neq \theta$. Therefore, $a \vee b \leq_\theta^e L$.
Case (ii): If $d \nleq b$, then clearly $d \nleq \theta$. Now $b \leq b \vee d \in [b, 1]$, and by Lemma 2.24 and by modular law, we have $\theta \leq b \neq (a \vee b) \wedge (b \vee d) = ((a \vee b) \wedge d) \vee b$, shows that $a \vee b \leq_\theta^e L$. □

Lemma 2.26. Let L be upper continuous and b be a θ-complement of a in L. If $c \in L$ is maximal such that $a \leq c$, $b \wedge c = \theta$, then c is maximal with respect to $a \leq_\theta^e c$.

Proof. Let $\mathcal{K} = \{y \in L : a \leq y, b \wedge y = \theta\}$. Since $a \in \mathcal{K}$, $\mathcal{K} \neq \emptyset$. By Zorn's lemma, $\mathcal{K}$ has a maximal element, say c. To show $a \leq_\theta^e c$, let $x \in [\theta, c]$ such that

$a \wedge x = \theta$. Now,

$$\begin{aligned} a \wedge (b \vee x) &= (a \wedge c) \wedge (b \vee x) \\ &= a \wedge (c \wedge (b \vee x)) \\ &= a \wedge ((c \wedge b) \vee x), \text{ since } x \leq c, \text{ by modularity} \\ &= a \wedge (\theta \vee x) \\ &= a \wedge x \\ &= \theta. \end{aligned}$$

Since b is θ-complement of a, we have $b \vee x = b$ implies $x \leq b$. So, $x = x \wedge b \leq c \wedge b = \theta$, implies $x \leq \theta$. Therefore, $x = \theta$. For the maximality, let $a \leq c'$, and $a \leq_\theta^e c'$ such that $c \leq c'$. Then by hypothesis, we have $b \wedge c' \neq \theta$. Therefore, $a \wedge (b \wedge c') \neq \theta$. But $a \wedge (b \wedge c') = (a \wedge b) \wedge c' = \theta \wedge c' = \theta$, a contradiction. □

Lemma 2.27. Let $\theta < a \in L$, where L is upper continuous. Then a is θ-closed if and only if a is a θ-complement.

Proof. Suppose a is a θ-complement of b in L. In a contrary, assume that $a \leq_\theta^e c$, for some $c \in L$. Since $a \leq c$, by maximality of a, we have $b \wedge c \neq \theta$. Moreover, since $a \leq_\theta^e c$ and $b \wedge c \in [\theta, c]$, we have $a \wedge (b \wedge c) \neq \theta$, whereas, $a \wedge (b \wedge c) = (a \wedge b) \wedge c = \theta \wedge c = \theta$, a contradiction. Conversely, since L is upper continuous, and $\theta < a \in L$, by Corollary 2.23, we have a has a θ-complement, say b'. That is, b' is maximal such that $b' \wedge a = \theta$. Now to show, a is θ-complement of b', let c be maximal with respect to $a \leq c$ and $b' \wedge c = \theta$. Then by Lemma 2.26, c is maximal with respect to $a \leq_\theta^e c$. But since a is θ-closed in L, we get $a = c$. Therefore, a is θ-complement of b'. □

Proposition 2.28. Let L be θ-complemented. For any $b, c \in L$, if $b \wedge c = \theta$, $b \vee c \leq_\theta^e L$, and c is θ-essentially closed, then c is a θ-complement of b.

Proof. Let $b \wedge c = \theta$, $b \vee c \leq_\theta^e L$ and c is θ-essentially closed. In view of Lemma 2.24, it is enough to show $b \vee c \leq_\theta^e [c, 1]$. In a contrary, suppose that $(b \vee c) \wedge d = \theta$, for $\theta \neq d \in [c, 1]$. Now, $(b \vee c) \wedge d = \theta \leq c$ and $c \leq (b \vee c) \wedge d$. Therefore, $(b \vee c) \wedge d = c$. Since c is θ-closed, there exists $x \in L$ such that $\theta < x < d$, and $c \wedge x = \theta$. Then,

$$\begin{aligned} \theta &= c \wedge x \\ &= ((b \vee c) \wedge d) \wedge x \\ &= (b \vee c) \wedge (d \wedge x) \\ &= (b \vee c) \wedge x, \end{aligned}$$

a contradiction to $(b \vee c) \leq_\theta^e L$. □

Notation 2.29. If $a \wedge b = \theta$ and $(a \vee b) \wedge c = \theta$, then $a \wedge (b \vee c) = \theta$.

Proof. Let $a \wedge b = \theta$ and $(a \vee b) \wedge c = \theta$. Then,

$$\begin{aligned} a \wedge (b \vee c) &\leq (a \vee b) \wedge (b \vee c) \\ &= ((a \vee b) \wedge c) \vee b, \text{ by modular law} \\ &= \theta \vee b \\ &= b. \end{aligned}$$

Hence $a \wedge (b \vee c) \leq a \wedge b = \theta$. Also, $\theta \leq a$, $\theta \leq b \leq (b \vee c)$, imply $\theta \leq a \wedge (b \vee c)$. Therefore, $a \wedge (b \vee c) = \theta$. □

Proposition 2.30. Let c, b be θ-complements of b, a respectively in L such that $a \leq c$. Then

(1) b is a θ-complement of c in L; and $b \vee c \leq_\theta^e [b, 1]$;
(2) $a \leq_\theta^e c$.

Proof. (1) Suppose b is maximal with respect to $b \wedge a = \theta$. Let $d \in L$ and $\theta \leq b < d$ such that $c \wedge d = \theta$. Then $a \wedge d \leq c \wedge d = \theta$. Also, since $\theta \leq a \wedge d$, we get $a \wedge d = \theta$, a contradiction to the maximality of b. Thus, b is θ-complement of c in L. Now, by Lemma 2.24, we get $b \vee c \leq_\theta^e [b, 1]$.

(2) To show, $a \leq_\theta^e c$, let $a \wedge d = \theta$, where $d \in [\theta, c]$. Now, $(a \vee d) \wedge b \leq c \wedge b = \theta$. Also, $\theta \leq a \leq (a \vee d)$, $\theta \leq b$, implies $\theta \leq (a \vee d) \wedge b$. Therefore, $(a \vee d) \wedge b = \theta$. Then by Note 2.29, we have $a \wedge (d \vee b) = \theta$. Now, by maximality of b, we get $d \vee b = b$. Therefore, $d \leq b$ and $d \leq b \wedge c = \theta$, shows that $d = \theta$.

□

Proposition 2.31. Let b be a θ-complement of a in L. If $\theta < c \leq_\theta^e L$, then $b \vee c \leq_\theta^e [b, 1]$.

Proof. Let $d \in [b, 1]$ such that $(b \vee c) \wedge d = \theta$. Then clearly, $(b \vee c) \wedge d = \theta \leq b$, and $b \leq (b \vee c) \wedge d$, implies $(b \vee c) \wedge d = b$. Now, by modular law $b = (b \vee c) \wedge d = b \vee (c \wedge d)$, and so $c \wedge d \leq b$. Then, $a \wedge (c \wedge d) \leq a \wedge b = \theta$. Also, $\theta \leq a$, $\theta \leq c \leq (c \wedge d)$ implies $c \wedge (a \wedge d) = a \wedge (c \wedge d) = \theta$. Since, $c \leq_\theta^e L$, we get $a \wedge d = \theta$. Then by the maximality of b, we get $d = b = \theta$, and shows $b \vee c \leq_\theta^e [b, 1]$. □

Lemma 2.32. Let $1 \in L$, $b < a$ in L and $a \leq_\theta^e [b, 1]$. Then $a \wedge c \leq_\theta^e [b \wedge c, c]$, for all $c \in L$.

Proof. Suppose $(a \wedge c) \wedge x = \theta$, where $x \in [b \wedge c, c]$. Now, taking the join with b on both side we get $[a \wedge (c \wedge x)] \vee b = \theta \vee b = \theta$, since $\theta \in [b, 1]$. By modular law, since $b < a$, we have $a \wedge [(c \wedge x) \vee b] = \theta$. Since $a \leq_\theta^e [b, 1]$ and $(c \wedge x) \vee b \in [b, 1]$, we get $(c \wedge x) \vee b = \theta$. Since $x \leq c$, $x \vee b = \theta$, and hence $x \leq \theta$. Also, $\theta \leq x$. Therefore, $x = \theta$, as desired. □

Lemma 2.33. [19] Let $\theta < b < a$ be in L. Then, $a \leq_\theta^e L$ and $b \leq_\theta^e [\theta, a]$ if and only if $b \leq_\theta^e L$.

Notation 2.34. Let x, y be elements of L. If $x \vee y \leq_\theta^e L$, then $x \vee y \in [\theta, 1]$.

Lemma 2.35. Let $1 \in L$. If L is θ-complemented, then for every $a \in L$, $[\theta, a]$ is also θ-complemented.

Proof. Suppose L is θ-complemented. Let $x \in [\theta, a] \subseteq L$. By Corollary 2.25, there exists $y \in L$ such that $x \wedge y = \theta$ and $x \vee y \leq_\theta^e L$. Then by Note 2.34, we have $x \vee y \leq_\theta^e [\theta, 1]$. Now, $a \wedge (x \wedge y) = a \wedge \theta = \theta$, implies $x \wedge (y \wedge a) = \theta$. Then, by Lemma 2.32, $(x \vee y) \wedge a \leq_\theta^e [(\theta \wedge a), 1 \wedge a] = [\theta, a]$. Since $x \leq a$, by modular law we get $x \vee (y \wedge a) \leq_\theta^e [\theta, a]$. Therefore, $[\theta, a]$ is complemented. □

Proposition 2.36. If $[\theta, a]$ is θ-complemented in L, for some $a \leq_\theta^e L$, then L is also θ-complemented.

Proof. Let $[\theta, a]$ be θ-complemented. For $x \in L$, $x \wedge a \in [\theta, a]$ has a θ-complement in $[\theta, a]$, say y. Then by Corollary 2.25, we have $y \wedge (x \wedge a) = \theta$, and $y \vee (x \wedge a) \leq_\theta^e [\theta, a]$. By Lemma 2.33, $y \vee (x \wedge a) \leq_\theta^e L$. Now to show $y \vee x \leq_\theta^e L$, let $z \in L$ such that $(y \vee x) \wedge b = \theta$. Then, $[(y \vee x) \wedge a] \wedge b \leq (y \vee x) \wedge b = \theta$. Also, since $\theta \leq b$, $\theta \leq [y \vee (x \wedge a)]$, implies $\theta \leq [y \vee (x \wedge a)] \wedge b$. Hence, $[y \vee (x \wedge a)] \wedge b = \theta$. Since $y \vee (x \wedge a) \leq_\theta^e L$, we get $b = \theta$. Thus $y \vee x \leq_\theta^e L$, shows that y is θ-complement of x in L. □

Proposition 2.37. Let L is θ-complemented and $1 \in L$. If a is a θ-complement in L, then $[a, 1]$ is also θ-complemented.

Proof. Suppose a is a θ-complement of b in L. Then, by Lemma 2.24, we have $a \wedge b = \theta$, and $a \vee b \leq_\theta^e [a, 1]$, and by Lemma 2.35, $[\theta, b]$ is also θ-complemented. Now, $[a, a \vee b] \cong [a \wedge b, b] = [\theta, b]$, is θ-complemented. That is, $[a, a \vee b]$ is θ-complemented. Therefore by Proposition 2.36, $[a, 1]$ is θ-complemented. □

Theorem 2.38. If b is a θ-complement of a in L and $\theta < c \leq_\theta^e L$, then b is a θ-complement of $a \wedge c$ in L.

Proof. Let b be a θ-complement of a in L. Then $a \wedge b = \theta$ and $a \vee b \leq_\theta^e L$. Clearly, $(a \wedge c) \wedge b = \theta$. Now let $d = a \vee b$ and $u = (a \wedge c) \vee b$. To show $u \leq_\theta^e L$, let $u \wedge y = \theta$, for some $y \in L$. Let $x = y \wedge d$. Now, $u \wedge x = u \wedge (y \wedge d) = (u \wedge y) \wedge d = \theta \wedge d = \theta$. Since $b \leq u$ by modular law, $u \wedge (b \vee x) = b \vee (u \wedge x) b \vee \theta = b$, and

$$\begin{aligned} \theta &= a \wedge b \\ &= a \wedge [u \wedge (b \vee x)] \\ &= a \wedge [(a \wedge c) \vee b] \wedge (b \vee x). \end{aligned}$$

Now since $a \wedge c \leq a$ and by modularity we get $\theta = [(a \wedge c) \vee (a \wedge b)] \wedge (b \vee x) = (a \wedge c) \wedge (b \vee x)$. Since $c \leq_\theta^e L$, we get $a \wedge (b \vee x) = \theta$. Then,

$$\begin{aligned} b &= b \vee \theta \\ &= b \vee [a \wedge (b \vee x)] \\ &= (b \vee x) \wedge (b \vee a), \text{ by modular law, } b \leq b \vee x \\ &= b \vee x, \end{aligned}$$

which implies $x \leq b$. Now $\theta = u \wedge x \geq b \wedge x = x$. Also, $\theta \leq x$, implies $x = \theta$. Since $d \leq_\theta^e L$, we get $y = \theta$. Thus, b is θ-complement of $a \wedge c$ in L.

□

Conclusions

We have defined the module theoretical concepts such as θ-complement, θ-closed and relative θ-complemented in a lattice. In a modular lattice, we have proved characterizations involving θ-complements with necessary illustrations. The results can be extended to study the dual aspects like supplements, superfluous and radicals etc. in a lattice. Possibly, one can study the concepts in hyperlattices, as the authors explored several hyperstructural aspects of lattices in [17, 14].

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