Global and Stochastic Analysis
Vol. 11 No. 3 (June, 2024)

Received: 12th April 2024 ***Revised: 27th May 2024*** ***Accepted: 30th May 2024***

SUM OF SECONDARY RANGE SYMMETRIC MATRICES

DIVYA SHENOY PURUSHOTHAMA*

Abstract. A necessary and sufficient condition for sum of secondary range symmetric matrices to be secondary range symmetric is given here. Also, the concept of parallel summable secondary range symmetric matrices and some of its characterizations are being explored.

1. Introduction

The concepts of secondary symmetric matrices is introduced by [4]. Vijayakumar et.al [2], [3], [12] defined the concept of secondary normal matrices and its characterizations. Based on this concept, secondary generalized inverses and its characterizations are obtained in [7]. Drazin theta inverse [10] is a generalized inverse of a matrix A, obtained by combining Drazin inverse as well as the secondary conjugate transpose of the matrix A. For more porperties and definitions involving secondary (conjugate) transpose, one can refer [10], [8], [9].

In this article, we give essential conditions for sum of secondary range symmetric matrices to be secondary symmetric. Also, the parallel summability of secondary range symmetric matrices are obtained. Below are some preliminary results from the literature.

2. Preliminaries

Throughout this article, $\mathbb{C}^{m\times n}$ represents the set of all matrices over complex field.

Definition 2.1. [1] A^- is a generalized inverse (g-inverse) of A if $AA^-A = A$.

Definition 2.2. [1] A matrix $A \in \mathbb{C}^{n\times n}$ is range symmetric (or EP) if $Null(A) = Null(A^*)$.

Definition 2.3. [7] $A^{\dagger_s}$ is a secondary generalized inverse of A if $AA^{\dagger_s}A = A$, $A^{\dagger_s}AA^{\dagger_s} = A$ and $AA^{\dagger_s}$, $A^{\dagger_s}A$ are secondary symmetric.

Definition 2.4. [2] Let $A \in \mathbb{C}^{n\times n}$. Then the conjugate secondary transpose of A denoted by A^θ and is defined as $A^\theta = \overline{A}^s = (c_{ij})$ where $c_{ij} = \overline{a}_{n-j+1,n-i+1}$.

Whenever we consider the matrix on set of real numbers, the secondary conjugate transpose A^θ reduces to secondary transpose A^S.
A relation connecting the transpose and secondary transpose of the matrix A is,

---

2000 *Mathematics Subject Classification.* Primary 15A09, 15A24; Secondary 65F05.
Key words and phrases. EP matrix, secondary transpose, secondary generalized inverse.

$A^S = VA^TV$ where V is a permutation matrix having unity in the secondary diagonal.

Definition 2.5. [11] A matrix $A \in \mathbb{R}^{n\times n}$ secondary range symmetric if and only if $Null(A) = Null(A^S)$

Theorem 2.6. [11] *Let $A \in \mathbb{R}^{n\times n}$. Then the following conditions are equivalent.*

(i) *A is secondary range symmetric*
(ii) *VA is range symmetric*
(iii) *AV is range symmetric*
(iv) $Null(A^*) = Null(AV)$
(v) $\mathcal{C}(A) = \mathcal{C}(A^S)$
(vi) *$A^S = PA = AQ$ where P and Q are some nonsingular matrices*
(vii) $\mathcal{C}(A^*) = \mathcal{C}(VA)$
(viii) $\mathcal{C}(A^*) \oplus Null(A) = \mathbb{C}^n$
(ix) $\mathcal{C}(A) \oplus Null(A^*) = \mathbb{C}^n$

3. Results

Lemma 3.1. *Let $A_1, A_2, ...A_k \in \mathbb{R}^{n\times n}$. If $A = \sum_{i=1}^{k} A_i$, then $A^S = \sum_{i=1}^{k} A_i^S$*

Proof. By definition $A_i^S = VA_i^*V$ for $i = 1, 2, ...k$.
To prove $A^S = \sum_{i=1}^{k} A_i^S$,

Given $A = \sum_{i=1}^{S} A_i$.
Now,

$$\begin{aligned} A^S &= V(A_1 + A_2 + ... + A_k)^*V \\ &= V(A_1^* + A_2^* + ... + A_k^*)V \\ &= A_1^S + A_2^S + ... + A_k^S \end{aligned}$$

Hence $A^S = \sum_{i=1}^{k} A_i^S$.

□

Lemma 3.2. *Let $A, B \in \mathbb{R}^{n\times n}$, then*

(i) $(AB)^S = B^S A^S$
(ii) $(A^S)^S = A$

Proof.

$$\begin{aligned} \text{By definition}(AB)^S &= V(AB)^*V \\ &= V(B^*A^*)V \\ &= (VB^*V)(VA^*V) \\ &= B^S A^S \end{aligned}$$

(ii) follows from (i) since $(A^S)^S = (VA^*V)^S = V(A^*)^SV = A$. □

Lemma 3.3. [5] *Let $A_1, A_2, ..., A_k \in \mathbb{C}^{n\times n}$ and let $A = \sum_{i=1}^{k} A_i$. Consider the following conditions.*

(i) $Null(A) \subseteq Null(A_i)$; $i = 1, 2, ..., k$

(ii) $Null(A) = \bigcap_{i=1}^{k} Null(A_i)$

(iii) $rank(A) = rank \begin{pmatrix} A_1 \\ A_2 \\ . \\ . \\ A_k \end{pmatrix}$

(iv) $\sum_{i=1}^{k} \sum_{i=1}^{n} A_i^* A_j = 0$

(v) $rank(A) = \sum_{i=1}^{k} rank(A_i)$

Then the following statement hold:
(a) (i), (ii), (iii) are equivalent.
(b) conditions (iv) implies (i) but the converse is not true.
(c) conditions (v) implies (i) but not the converse.

Theorem 3.4. *Let $A_i (i = 1, 2, ..., k)$ be secondary range symmetric. Assume any one of the condition of previous lemma holds. Then $A = \sum_{i=1}^{k} A_i$ is secondary range symmetric.*

Proof. Since each A_i is secondary range symmetric, by definition 2.5, $Null(A_i) = Null(A_i^S)$ for each $i = 1, 2, ..., k$. By the given condition

$$Null(A) \subseteq Null(A_i)$$

we get

$$Null(A) \subseteq \bigcap_{i=1}^{k} Null(A_i) = \bigcap_{i=1}^{k} Null(A_i^S)$$

Now,

$$\begin{aligned} x \in Null(A) \subseteq \bigcap_{i=1}^{k} Null(A_i^S) &\implies x \in Null(A_i^S),\ for\ i = 1\ to\ k \\ &\implies A_i^S x = 0,\ for\ i = 1\ to\ k \\ &\implies (A_1^S + A_2^S + ... + A_k^S)x = 0 \\ &\implies A^S x = 0 \end{aligned}$$

$\bigcap_{i=1}^{k} Null(A_i^S) \subseteq Null(A^S)$

$Null(A) \subseteq \bigcap_{i=1}^{k} Null(A_i^S) \subseteq Null(A^S)$ and $\rho(A) = \rho(A^S)$ implies $Null(A) = Null(A^S)$. Thus $A = \bigcap_{i=1}^{k} A_i$ is secondary range symmetric. □

The converse of the above theorem is not true.
Let $A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.
Assume $A = A_1 + A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Here, A_1, A_2 and A are secondary range symmetric matrices.
But $Niull(A_1 + A_2) \nsubseteq Null(A)$.
If A and B are secondary range symmetric matrices, then by Theorem 2.6 $A^S = P_1 A$ and $B^S = P_2 A$ where P_1 and P_2 are non singular square matrices. If $P_1 = P_2$, say P, then $A^S + B^S = P(A + B) = (A + B)^S$ which implies $A + B$ is secondary range symmetric.

Theorem 3.5. *Let A and B be secondary range symmetric matrices, then $A^S = P_1 A$ and $B^S = P_2 B$ such that $P_1 - P_2$ is a non singular matrix. Then $A + B$ is secondary range symmetric iff $Null(A + B) \subseteq Null(B)$.*

Proof. Since $A^S = P_1 A$ and $B^S = P_2 B$, from theorem 2.6, the matrices A and B are secondary range symmetric. Since $Null(A + B) \subseteq Null(B)$, we can see that $Null(A + B) \subseteq Null(A)$. Hence by theorem 3.4, $A + B$ is secondary range symmetric.

Conversly, let us assume that $A + B$ is secondary range symmetric. Now by Theoem 2.6, $(A + B)^S = P(A + B)$
which implies

$$P(A + B) = A^S + B^S = P_1 A + P_2 B$$

$$(P_1 - P)A = (P - P_2)B$$

$$EA = FB$$

where $E = P_1 - P$ and $F = P - P_2$ such that $E + F = P_1 - P_2$.

$$EA + FA = EB + FA$$

$$(E + F)A = F(A + B)$$

By hypothesis $E + F = H_1 - H_2$ is non singular. $Null(A+B) \subseteq Null[F(A+B)] = Null[(E + F)A] = Null(A)$. Similarly we can see that $Null(A + B) \subseteq Null(B)$. Thus $A + B$ is secondary range symmetric implies $Null(A + B) \subseteq Null(A)$ and $Null(B)$. Hence the Theorem. □

4. Parallel summable secondary range symmetric matrices

Definition 4.1. [6] Let $A, B \in \mathbb{C}^{m\times n}$ matrices. If $Null(A + B) \subseteq Null(B)$ and $Null(A + B)^* \subseteq Null(B^*)$ or $Null(A + B) \subseteq Null(A)$ and $Null(A + B)^* \subseteq Null(A^*)$ then the matrices A and B are said to be parallel summable.

Definition 4.2. [6] If two matricesA and B are parallel summable, then the parallel sum of is denoted by $A : B$ is defined as $A : B = A(A + B)^- B$.

Lemma 4.3. *Let A and B be matrices. Then $Null(A^*) \subseteq Null(B^*)$ if and only if $Null(A^S) \subseteq Null(B^S)$*

Proof. Assume that $Null(A^*) \subseteq Null(B^*)$

$$\begin{aligned} x \in Null(A^S) &\implies A^S x = 0 \\ &\implies VA^*Vx = 0 \\ &\implies A^*Vx = 0 \\ &\implies A^*y = 0 \text{ where } y = Vx \\ &\implies y \in Null(A^*) \subseteq Null(B^*) \\ &\implies B^*y = 0 \\ &\implies B^*Vx = 0 \\ &\implies B^S x = 0 \\ &\implies x \in Null(B^S) \end{aligned}$$

Thus $Null(A^S) \subseteq Null(B^S)$.
Conversly, let us assume that $Null(A^S) \subseteq Null(B^S)$. We need to show that $Null(A^*) \subseteq Null(B^*)$.
Let us choose

$$\begin{aligned} x \in Null(A^*) &\implies A^*x = 0 \\ &\implies (VA^*V)Vx = 0 \\ &\implies A^S Vx = 0 \\ &\implies A^S y = 0 \\ &\implies y \in Null(A^S) \subseteq Null(B^S) \\ &\implies B^S y = 0 \\ &\implies VB^*VVx = 0 \\ &\implies VB^*x = 0 \\ &\implies B^*x = 0 \\ &\implies x \in Null(B^*) \end{aligned}$$

Thus $Null(A^*) \subseteq Null(B^*)$. Hence the Lemma. □

Definition 4.4. A pair of matrices A and B are said to be secondary parallel summable if $Null(A+B) \subseteq Null(B)$ and $Null(A+B)^S \subseteq Null(B^S)$ or equivalently $Null(A+B) \subseteq Null(A)$ and $Null(A+B)^S \subseteq Null(A^S)$

5. Properties

Let A and B be a pair of secondary parallel summable matrices.

(i) $A : B$
(ii) A^S and B^S are secondary parallel summable and $(A : B)^S = A^S : B^S$
(iii) If C is a non singular matrix, then CA and CB are parallel summable and $CA : CB = C(A : B)$
(iv) $Null(A : B) = Null(A) + Null(B)$

Proof. Here we need to prove only (ii), A and B are secondary range symmetric, A^S and B^S are range symmetric follows from Lemma 3.1 and 3.2.

$$\begin{aligned} A^S : B^S &= A^S(A^S + B^S)^- B^S \\ &= A^S[(A+B)^S]^- B^S \\ &= A^S[(A+B)^-]^S B^S \\ &= [B(A+B)^- A]^S \\ &= [A(A+B)^- B]^S \\ &= [A : B]^S \end{aligned}$$

□

Lemma 5.1. *Let A and B be secondary range symmetric matrices. Then A and B are parallel summable secondary range symmetric if and only if $Null(A+B) \subseteq Null(A)$*

Proof. A and B are parallel summable, by definition 4.1 it follows that $Null(A+B) \subseteq Null(A)$.
Conversely, if $Null(A+B) \subseteq Null(A)$, then $Null(A+B) \subseteq Null(B)$. Since A and B are secondary range symmetric, $A+B$ is secondary range symmetric by Theorem 3.4. Hence $Null(A+B) = Null(A+B)^S$ and $Null(A+B) \subseteq Null(A)$ implies $Null(A+B)^S \subseteq Null(A^S)$. Then by Definition 4.1, A and B are parallel summable secondary range symmetric. Hence the proof. □

If A or B is not secondary range symmetric matrix, the above lemma is not true. Consider the following example:
Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A^S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Here A is secondary range symmetric matrix.
Also, let $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B^S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. It is clear that B is not secondary range symmetric.
Now, $A+B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $(A+B)^S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Here $Null(A+B) \subseteq Null(A)$, but $Null(A+B)^S \nsubseteq Null(A^S)$ and $Null(B^S)$. Hence A and B are not secondary parallel summable.

Theorem 5.2. *Let A and B be parallel summable secondary range symmetric. Then $A : B$ and $A+B$ are secondary range symmetric.*

Proof. Since A and B are parallel summable secondary range symmetric, $Null(A+B) \subseteq Null(A)$ and $Null(A+B) \subseteq Null(B)$, follows from above lemma. Now the fact that $A+B$ is secondary range symmetric follows from Theorem 3.4. $A : B$ is secondary range symmetric runs as follows.

$$\begin{aligned} Null(A:B)^S &= Null(A^S:B^S) \\ &= Null(A^S)+Null(B^S) \\ &= Null(A)+Null(B) \\ &= Null(A:B) \end{aligned}$$

Thus $A:B$ is secondary range symmetric whenever A and B are parallel summable secondary range symmetric. Hence the theorem. □

6. Conclusion

In general, the sum of two secondary range symmetric matrices need not be secondary range symmetric. We obtained a necessary and sufficient condition for sum of secondary range symmetric matrices to be secondary range symmetric. Similary, the essential conditions for product of secondary range symmetric matrices to be secondary range symmetric can be a topic for further research.

References

1. Ben-Israel. A, Greville. T. N. E, Generalized Inverses and Applications, Wiley,1974.
2. Krishnamoorthy. S, Vijayakumar. R, Some characteristics on s-normal matrices,International Journal of Computational and Applied mathematics, Vol.4, No.1,49- 53, 2009.
3. Krishnamoorthy. S, Vijayakumar. R, Some equivalent conditions on s-normal matrices,International Journal of Contemporary Mathematical Sciences, Vol.4, No.29,1449- 1454, 2009.
4. Lee. A, Secondary symmetric, Skew symmetric and orthogonal matrices, Periodica Mathematica Hungariva Vol. 7 (1), (1976), pp. 63–70
5. Meenakshi. A. R and Krishnaswamy. D, *On sum of range symmetric matrices in Minkowski space*, The Bulletin of the Malaysian Mathematical Society Series 25(2), 2002, pp. 137 -148.
6. Rao. C.R, Mitra S.K, *Generalized Inverse of Matrices and its Application*, Wiley and Sons, New York, 1971.
7. Savitha. V, Shenoy. D. P, Umashankar. K and Bapat. R. B, "Secondary transpose of a matrix and generalized inverses", Journal of Algebra and its applications, 23(3), Article No. 2450052, 2024, https://doi.org/10.1142/S021949882450052X.
8. Shenoy. D.P, *Outer theta and theta outer inverses*, IAENG International Journal of Applied Mathematics, 52(4), 2022, pp. 1020-1024.
9. Shenoy. D. P, *Drazin-Theta inverse for rectangular matrices*, IAENG International Journal of Computer Science, 50(4), 2023, pp. 1517-1521
10. Shenoy. D. P, *Drazin theta and theta Drazin matrices*, Numerical Algebra, Control and Optimization, 14(2),2024, pp. 273-283. doi: 10.3934/naco.2022023
11. Shenoy. D. P, *Secondary range symmetric matrices*, Shenoy D. Secondary range symmetric matrices [version 1; peer review: 1 approved]. F1000Research 2024, 13:112, https://doi.org/10.12688/f1000research.144171.1.
12. Vijayakumar. R, s-g inverse of s-normal matrices, International Journal of Mathematics Trends and Technology, Vol.39(4),(2016), pp. 240 -244

Divya Shenoy Purushothama: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India - 576104.
Email address: divya.shenoy@manipal.edu