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SPECTRAL ANALYSIS OF ARITHMETIC FUNCTION SIGNED GRAPHS

HOWIDA ADEL ALFRAN, R. RAJENDRA, P. SIVA KOTA REDDY*, R. KEMPARAJU, AND SAMI H. ALTOUM

Abstract. In this paper, we discuss spectrum and energy of Arithmetic function signed graphs with respect to an arithmetical function and present some results.

1. Introduction

A signed graph can be created by adding a sign to each edge of a simple graph. In 1953, Harary presented the idea of signed graphs [4] in relation to a few social psychology issues. In 1982, Zaslavsky introduced matroids of signed graphs [36]. Signed graphs are very important as they are used in the study of complex systems of computer models and sociology. For new notions in signed graphs, we suggest the reader to refer the papers [13, 17–21, 24–34]. Given a signed graph one can define a $(0, +1, -1)$-matrix called adjacency matrix. A signed graph is completely described by its adjacency matrix. Many researchers have recently focused a great deal of emphasis on the study of signed graph spectra [See [2, 6]]. Germina, Hameed and Zaslavsky [3] introduced the concept of energy of signed graphs.

The greatest common divisor of two numbers r and s is denoted by $\gcd(r, s)$ or (r, s). We assume that $\mathfrak{A}$ is a finite group. The concept of order prime graph $OP(\mathfrak{A})$ of a finite group $\mathfrak{A}$ was introduced by M. Sattanathan and R. Kala [22]. The order prime graph $OP(\mathfrak{A})$ of $\mathfrak{A}$ is the graph with vertex set $V(OP(\mathfrak{A})) = \mathfrak{A}$ and any two different vertices a and b are adjacent in $OP(\mathfrak{A})$ if and only if $(o(a), o(b)) = 1$. R. Rajendra and P. S. K. Reddy [7, 8] proposed the idea of a general order prime graph, taking into consideration the commuting property of elements in finite non-abelian groups. Sathyanaryana et al. [23] proposed the idea of a prime graph of an associative ring R and examined its features. Rajendra et al. [12] demonstrated that, in the case when n is a prime, the prime graph of the ring $\mathbb{Z}_n$ is equal to the order prime graph defined in [22] of the additive group $\mathbb{Z}_n$, where n denotes a positive integer. Rajendra et al. [10] introduced the idea of set-prime graph $G_S(\mathfrak{A})$ with regard to a set S of positive integers. Order prime and general order prime graphs are special cases of set-prime graphs.

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*Corresponding author.

As an another generalization of an order prime graph, Rajendra et al. [14] defined the arithmetic function graph of a finite group with respect to an arithmetical function:

Definition 1.1 (Arithmetic function graph [14])**.** The Arithmetic function graph $G_h(\mathfrak{A})$ of $\mathfrak{A}$ with respect to an arithmetical function h is defined as a graph with vertex set $V(G_h(\mathfrak{A})) = \mathfrak{A}$ and two vertices a and b are adjacent in $G_h(\mathfrak{A})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$.

In [14], the authors observed that the order prime graph of a finite group is nothing but the arithmetic function graph with respect to the Euler's ϕ-function. Also, the authors investigated some results related to diameter, dominating sets, planarity and isomorphism of arithmetic function graphs of finite groups. The eigenvalues and energy of the arithmetic function graphs of finite groups are discussed in [15].

An ordered pair $\mathcal{S} = (G, \sigma)$ is referred to as a signed graph [4, 36] when $G = (V, E)$ is a graph known as the underlying graph of $\mathcal{S}$ and $\sigma : E \to \{+, -\}$ is a function. An edge with end vertices u and v in a graph is denoted by uv or $\overline{(u, v)}$.

If there are an even number of negative edges in each cycle of a signed graph $\mathcal{S} = (G, \sigma)$, then $\mathcal{S}$ is balanced [4]. In contrast, if the product of the signs of the edges on each cycle of $\mathcal{S}$ is positive, then a signed graph is said to be balanced.

If each and every edge in a signed graph has the same sign, the graph is said to be homogenous; if not, it is said to be heterogeneous. The signed graph that results from reversing the signs on the edges of a signed graph $\mathcal{S}$ is called its negation of $\mathcal{S}$ and is denoted by $\neg\mathcal{S}$.

A graph can be thought of as an all-positive signed graph, and signed graphs are a generalization of graphs in this sense. A signed graph's sign is equal to the product of the signs of its edges. If the underlying undirected graphs of two signed graphs are isomorphic and the signs on the edges are retained, then the two signed graphs are isomorphic.

The concept of order prime signed graph of a finite group was introduced by Rajendra and Reddy [9]:

Definition 1.2 (Order prime signed graph [9])**.** The order prime signed graph $OPS(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is the signed graph $((OP(\mathfrak{A}), \sigma)$, where the function $\sigma : E(OP(\mathfrak{A})) \to \{+, -\}$ is given by

$$\sigma(\overline{(a, b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

The concept of arithmetic function signed graph of a finite group with respect to an arithmetic function was introduced by T. Kim et al [5]:

Definition 1.3 (Arithmetic function signed graph [5])**.** The arithmetic function signed graph $SG_h(\mathfrak{A})$ of a finite group $\mathfrak{A}$ is the signed graph $(G_h(\mathfrak{A}), \sigma)$ where the function $\sigma : E(G_h(\mathfrak{A})) \to \{+,-\}$ is given by

$$\sigma(\overline{(a,b)}) = \begin{cases} +, & \text{if } ab = ba; \\ -, & \text{otherwise.} \end{cases}$$

The following definition and theorem can be found in [5]:

Definition 1.4 ([5])**.** Let f be an arithmetic function. A group $\mathfrak{A}$ is said to be f-balanced if the signed graph $SG_f(\mathfrak{A})$ is balanced; otherwise we say that $\mathfrak{A}$ is f-unbalanced.

Theorem 1.5 ([5])**.** *(i) Every finite abelian group is f-balanced for any arithmetic function f.*
(ii) A non-abelian group of prime power order is f-balanced for any multiplicative function f.
(iii) A non-abelian group whose order is not a prime power, is f-unbalanced for any multiplicative function f.
(iv) A finite non-abelian group is f-unbalanced for any completely multiplicative function f.

In section 2, we recall the concepts of spectrum and energy of signed graphs. We discuss spectrum and energy of arithmetic function signed graphs of finite groups and we present some results in section 3.

2. Spectrum and Energy of Signed Graphs

Let $\mathcal{S} = (G,\sigma)$ be a signed graph, where G is the underlying graph of $\mathcal{S}$ with vertex set $V(G) = \{u_1, u_2, \cdots, u_n\}$ and $\sigma : E(G) \to \{+,-\}$ is a signing function. The adjacency matrix of the signed graph $\mathcal{S}$, denoted by $A(\mathcal{S})$, is the square matrix (x_{ij}) where

$$x_{ij} = \begin{cases} 1, & \text{if } u_iu_j \in E(G) \text{ and } \sigma(u_iu_j) = +; \\ -1, & \text{if } u_iu_j \in E(G) \text{ and } \sigma(u_iu_j) = -; \\ 0, & \text{otherwise.} \end{cases}$$

An adjacency matrix completely describes a signed graph.

The eigenvalues of the adjacency matrix $A(\mathcal{S})$ of $\mathcal{S}$ are called the eigenvalues of $\mathcal{S}$. The set of eigenvalues of $\mathcal{S}$ is also called the spectrum of $\mathcal{S}$ and we denote this by $Spec(\mathcal{S})$.

If the distinct eigenvalues of $\mathcal{S}$ are $\gamma_1 > \gamma_2 > \cdots > \gamma_k$ and their multiplicities are $s_1, s_2, \ldots, s_k$, respectively, then the spectrum of $\mathcal{S}$ is

$$Spec(\mathcal{S}) = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix}.$$

We recall the following theorem by B.D. Acharya [1], which provides a spectral criteria for balance in signed graphs:

Theorem 2.1 (B.D. Acharya [1])**.** *A signed graph $\mathcal{S} = (G,\sigma)$ is balanced if and only if $Spec(\mathcal{S}) = Spec(G)$.*

The idea of the energy of a signed graph was introduced by Germina, Hameed and Zaslavsky [3]. The energy of a signed graph $\mathcal{S}$, denoted by $\mathcal{E}(\mathcal{S})$, is the sum of the absolute values of its eigenvalues. That is, if $\gamma_1, \gamma_2, \ldots, > \gamma_n$ are the eigenvalues of $\mathcal{S}$, then

$$\mathcal{E}(\mathcal{S}) = \sum_{i=1}^{n} |\gamma_i|.$$

3. Spectrum and Energy of Arithmetic Function Signed Graphs

In this section, we discuss spectrum and energy of arithmetic function signed graphs of finite groups and we present some results. The following result is immediate from the Definition 1.4 and the Theorem 2.1.

Theorem 3.1. *Let $\mathfrak{A}$ be a finite group and f be a multiplicative function. Then, $\mathfrak{A}$ is f-balanced if and only if $Spec(SG_f(\mathfrak{A})) = Spec(G_f(\mathfrak{A}))$.*

The following definition of the order prime signed energy of a finite group was introduced by R. Rajendra and P.S.K. Reddy [16].

Definition 3.2 (Order prime signed energy of a finite group [16])**.** Let $\mathfrak{A}$ be a group of finite order. The order prime signed energy of the group $\mathfrak{A}$, denoted by $\mathcal{OPSE}(\mathfrak{A})$, is defined as the energy of the order prime signed graph $OPS(\mathfrak{A})$. That is, $\mathcal{OPSE}(\mathfrak{A}) = \mathcal{E}(OPS(\mathfrak{A}))$.

Given an arithmetic function and a finite group $\mathfrak{A}$, we define f_s-energy of a finite group as follows:

Definition 3.3 (f_s-energy of a finite group)**.** Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. The f_s-energy of $\mathfrak{A}$ is the energy of the signed graph $SG_f(\mathfrak{A})$. The f_s-energy of $\mathfrak{A}$ is denoted by $f_{se}(\mathfrak{A})$.

Observation 3.4. By the Definitions 3.2 and 3.3, it follows that, for a finite group $\mathfrak{A}$, the order prime signed energy of $\mathfrak{A}$ is nothing but ϕ_s-energy of $\mathfrak{A}$. That is,

$$\mathcal{OPSE}(\mathfrak{A}) = \phi_{se}(\mathfrak{A}).$$

Proposition 3.5. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. If $\mathfrak{A}$ is f-balanced, then $f_{se}(\mathfrak{A}) = \mathcal{E}_f(\mathfrak{A})$.*

Proof. The result follows by Definition 3.3 and Theorem 3.1. □

Proposition 3.6. *Let f be a completely multiplicative arithmetic function and $\mathfrak{A}$ be a finite group of order n. Then*

(i) If $\mathfrak{A}$ is abelian, then $f_{se}(\mathfrak{A}) = 2n - 2$.

(ii) If $\mathfrak{A}$ is non-abelian, then $f_{se}(\mathfrak{A}) = \mathcal{E}((K_n, \sigma))$, where σ is the signing function given in the Definition 1.3.

Proof. Since f is a completely multiplicative arithmetic function, $G_f(\mathfrak{A}) \cong K_n$.

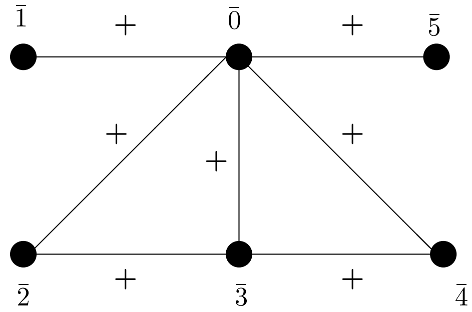
(i) If $\mathfrak{A}$ is abelian, then $SG_f(\mathfrak{A}) \cong K_n^+$ and so

$$f_{se}(\mathfrak{A}) = \mathcal{E}(K_n^+) = \mathcal{E}(K_n) = 2n - 2.$$

(ii) If $\mathfrak{A}$ is non-abelian, then $SG_f(\mathfrak{A}) = (G_f(\mathfrak{A}), \sigma) \cong (K_n, \sigma)$, where σ is the signing function given in the Definition 1.3. Hence, $f_{se}(\mathfrak{A}) = \mathcal{E}((K_n, \sigma))$.

□

Example 3.7. Consider the group $\mathbb{Z}_6$ under the operation addition modulo 6. The corresponding arithmetic function-ϕ signed graph $SG_\phi(\mathbb{Z}_6)$ (i.e., order prime signed graph $OPS(\mathbb{Z}_6)$) and its adjacency matrix are exhibited in Figure 1.



$SG_\phi(\mathbb{Z}_6)$

$$A(SG_\phi(\mathbb{Z}_6)) = \begin{array}{c} \\ \bar{0} \\ \bar{1} \\ \bar{2} \\ \bar{3} \\ \bar{4} \\ \bar{5} \end{array} \begin{array}{c} \begin{array}{cccccc} \bar{0} & \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

FIGURE 1. The signed graph $SG_\phi(\mathbb{Z}_6) = OPS(\mathbb{Z}_6)$ and its adjacency matrix

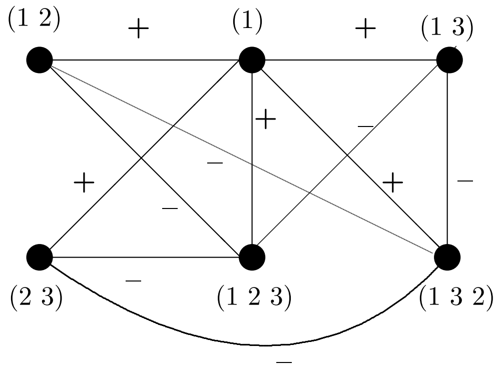
The spectrum of $SG_\phi(\mathbb{Z}_6)$ is

$$Spec(SG_\phi(\mathbb{Z}_6)) = \begin{pmatrix} 0 & -2 & -1.3429231 & 0.5293166 & 2.8136065 \\ 2 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The eigenvalues are computed using Scilab software. The ϕ_s-energy of $\mathbb{Z}_6$) is

$$\begin{aligned} \phi_{es}(\mathbb{Z}_6) = \mathcal{OPSE}(\mathbb{Z}_6)) &= |0| + |0| + |-2| + |-1.3429231| \\ &+ |0.5293166| + |2.8136065| \\ &= 6.6858462. \end{aligned}$$

Example 3.8. Consider the permutation group S_3 of 3 symbols.



$SG_\phi(S_3)$

$$A(SG_\phi(S_3)) = \begin{array}{c} \\ (1) \\ (1\ 2) \\ (1\ 3) \\ (2\ 3) \\ (1\ 2\ 3) \\ (1\ 3\ 2) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (1\ 2) & (1\ 3) & (2\ 3) & (1\ 2\ 3) & (1\ 3\ 2) \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

FIGURE 2. The signed graph $SG_\phi(S_3)$ and its adjacency matrix

The arithmetic function-ϕ signed graph $SG_\phi(S_3)$ (i.e., order prime signed graph $OPS(S_3)$) and its adjacency matrix are exhibited in Figure 2.

The spectrum of $SG_\phi(S_3)$ is

$$Spec(SG_\phi(S_3)) = \begin{pmatrix} 0 & -3.7664355 & 1.2828239 & 2.4836116 \\ 3 & 1 & 1 & 1 \end{pmatrix}.$$

The eigenvalues are computed using Scilab software. The ϕ_s-energy of $\mathbb{Z}_6$) is

$$\begin{aligned} \phi_{es}(S_3) &= |0| + |0| + |0| + |-3.7664355| + |1.2828239| + |2.4836116| \\ &= 7.532871. \end{aligned}$$

The following three results are immediate from [35]:

Theorem 3.9. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. If f_s-energy of $\mathfrak{A}$ is a rational number, then it is an even integer.*

Proof. By [35, Theorem 3], the energy of a signed graph is rational, then it is an even integer. Hence, by the Definition 3.3, the result follows. □

Theorem 3.10. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. The f_s-energy of $\mathfrak{A}$ can never be a r-th root of a positive rational number which is not an integer.*

Proof. By [35, Theorem 4], if the energy of a signed graph can never be a r-th root of a positive rational number which is not an integer. Hence, by the Definition 3.3, the result follows. □

Theorem 3.11. *Let f be an arithmetic function and $\mathfrak{A}$ be a finite group. The f_s-energy of $\mathfrak{A}$ cannot be of the form $2^{\frac{s}{t}}a^{\frac{1}{t}}$, where a is odd, $t \geq 1$ and $0 \leq s \leq t-1$.*

Proof. The result follows by [35, Theorem 5] and Definition 3.3. □

The following result gives spectra and energy of the arithmetic function signed graph of a group of prime power order with respect to a multiplicative function.

Theorem 3.12. *If h is a multiplicative function, then for any group of prime power order p^η, where η is a positive integer.*

(i) $SG_h(\mathfrak{A}) = K^+_{1,p^\eta-1}$.

(ii) $Spec(SG_h(\mathfrak{A})) = \begin{pmatrix} \sqrt{p^\eta-1} & 0 & -\sqrt{p^\eta-1} \\ 1 & p^\eta-2 & 1 \end{pmatrix}$,

and hence

$$h_{se}(\mathfrak{A}) = 2\sqrt{p^\eta-1}.$$

Proof. (i) Suppose that h is a multiplicative function and $\mathfrak{A}$ be a group of prime power order p^η. Then $G_h(\mathfrak{A})$ is a star $K_{1,p^\eta-1}$ with center e, the identity element in $\mathfrak{A}$. Since e commutes with every other element in $\mathfrak{A}$, it follows that, all the edges of $SG_h(\mathfrak{A})$ are assigned '+' sign in $SG_h(\mathfrak{A})$. Hence,

$$SG_h(\mathfrak{A}) = G_h(\mathfrak{A})^+ = K^+_{1,p^\eta-1}.$$

(ii) We have $A(SG_h(\mathfrak{A})) = A(K^+_{1,p^\eta-1}) = A(K_{1,p^\eta-1})$. This implies that

$$\begin{aligned} Spec(SG_h(\mathfrak{A})) &= Spec(K_{1,p^\eta-1}) \\ &= \begin{pmatrix} \sqrt{p^\eta-1} & 0 & -\sqrt{p^\eta-1} \\ 1 & p^\eta-2 & 1 \end{pmatrix}. \end{aligned}$$

Hence

$$h_{se}(\mathfrak{A}) = |\sqrt{p^\eta-1}| + (p^\eta-2)\cdot|0| + |-\sqrt{p^\eta-1}| = 2\sqrt{p^\eta-1}.$$

□

The following theorem characterizes the f-balanced non-abelian groups with respect to a multiplicative function f.

Theorem 3.13. *For a non-abelian finite group $\mathfrak{A}$ and a multiplicative function f, the following statements are equivalent:*

(i) $\mathfrak{A}$ *is of prime power order.*

(ii) $\mathfrak{A}$ *is f-balanced.*

(iii) $Spec(SG_f(\mathfrak{A})) = Spec(G_f(\mathfrak{A}))$.

Proof. $(i) \implies (ii)$: Suppose that $\mathfrak{A}$ is of prime power order. Then by Theorem 1.5, $\mathfrak{A}$ is f-balanced.

$(ii) \implies (i)$: Suppose that $\mathfrak{A}$ is f-balanced. Then $SG_f(\mathfrak{A})$ is balanced and hence every cycle in $SG_f(\mathfrak{A})$ possesses an even number of negative edges. We claim that $a, b \in \mathfrak{A}$ are not adjacent in $SG_f(\mathfrak{A})$ if $ab \neq ba$. For, if $a, b \in \mathfrak{A}$ are adjacent in $SG_f(\mathfrak{A})$ when $ab \neq ba$, then e, a, b form a 3-cycle in $SG_f(\mathfrak{A})$ with $\sigma(\overline{(e,a)}) = +, \sigma(\overline{(e,b)}) = +$ and $\sigma(\overline{(a,b)}) = -$. This implies that there is a 3-cycle in $SG_f(\mathfrak{A})$ with odd number of negative edges and consequently, $SG_f(\mathfrak{A})$ is unbalanced, which contradicts the fact that $SG_f(\mathfrak{A})$ is balanced. Therefore our claim holds. Consequently, $SG_f(\mathfrak{A}) = G_f(\mathfrak{A})^+ \cong K^+_{1,n-1}$, where $n = o(\mathfrak{A})$. This shows that $G_f(\mathfrak{A}) \cong K_{1,n-1}$.

Now, we shall establish that n is a power of a prime. Without loss of generality, we assume that $n = p^{\eta_1} q^{\eta_2}$, where p, q are primes, $p \neq q$ and η_1, η_2 are positive integers. Since p and q are divisors of $n = o(\mathfrak{A})$, by Cauchy's theorem for finite groups, there exist elements say, a, b in $\mathfrak{A}$ such that $|a| = p$ and $|b| = q$. Since $p \neq q$, we have $(p, q) = 1$. Since f is multiplicative $(p, q) = 1$ implies that $f(pq) = f(p)f(q)$ i.e., $f(|a||b|) = f(|a|)f(|b|)$. This implies that a, b are adjacent in $G_f(\mathfrak{A})$ and consequently, the vertices e, a, b form a 3-cycle in $G_f(\mathfrak{A})$. So, the graph $G_f(\mathfrak{A})$ is not a tree, contradicting the fact that $G_f(\mathfrak{A}) \cong K_{1,n-1}$. Thus, n is a power of a prime.

$(ii) \iff (iii)$: Follows from the Theorem 3.1.This ends the proof. □

The Theorem 3.13 can be stated in the notion of order prime signed graphs (by taking $f = \phi$) as follows:

Theorem 3.14. *For a non-abelian finite group $\mathfrak{A}$, the following statements are equivalent:*

(i) $\mathfrak{A}$ is of prime power order.
(ii) $OPS(\mathfrak{A})$ is balanced.
(iii) $Spec(OPS(\mathfrak{A})) = Spec(OP(\mathfrak{A}))$.

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Howida Adel AlFran: Department of Mathematics, AL-Leith University College, Umm Al-Qura University, Kingdom of Saudi Arabia.
Email address: hafran@uqu.edu.sa

R. Rajendra: Department of Mathematics, Field Marshal K. M. Cariappa College (A Constituent College of Mangalore University), Madikeri-571 201, India.
Email address: rrajendrar@gmail.com

P. Siva Kota Reddy: Department of Mathematics, JSS Science and Technology University, Mysuru-570 006, India
Email address: pskreddy@jssstuniv.in; pskreddy@sjce.ac.in

R. Kemparaju: Department of Mathematics, Government First Grade College, T. Narasipura-571 124, India.
Email address: kemps007@gmail.com

Sami H. Altoum: Department of Mathematics, AL-Qunfudhah University College, Umm Al-Qura University, Kingdom of Saudi Arabia
Email address: shtoum@uqu.edu.sa