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NORM PRINCIPLES FOR $\mathrm{GO}(q)$ AND $\mathrm{Spin}(q)$

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ABSTRACT. The aim of this article is to discuss the proof that the norm principle holds for the group of similitudes $\mathrm{GO}(q)$ and the spinor group $\mathrm{Spin}(q)$ for a quadratic form q defined over F.

1. Introduction:

In this paper, we explore norm principles for connected reductive algebraic groups defined over a field F, where char F is different from 2. Consider an abelian algebraic group T over the field F which is linear and let K/F be a finite field extension which is separable. The *norm homomorphism* $N_{K/F}$, is defined as

$$N_{K/F} : T(K) \to T(F)$$

by sending $t \mapsto \prod_\gamma \gamma(t)$, where $\gamma \in \frac{\mathrm{Gal}(F^{\mathrm{sep}}/F)}{\mathrm{Gal}(F^{\mathrm{sep}}/K)}$ and F^{sep} denotes the separable closure of F. If $T = \mathbb{G}_m$, then $N_{K/F}$ is the usual field norm $N_{K/F} : K^\times \to F^\times$.

Suppose $G(F)$ is a connected algebraic group over F which is linear, and $T(F)$ is a commutative algebraic group which is also linear over F. Let us consider an algebraic group homomorphism $\phi : G(F) \to T(F)$. For a separable field extension K/F that is finite, let ϕ_K be the algebraic group homomorphism from $G(K)$ to $T(K)$. Let us look into the following diagram:

$$\begin{array}{ccc} G(K) & \xrightarrow{\phi_K} & T(K) \\ & & \Big\downarrow{\scriptstyle N_{K/F}} \\ G(F) & \xrightarrow{\phi} & T(F) \end{array}$$

We say ϕ satisfies the norm principle if

$$N_{K/F}(\phi_K(G_K)) \subseteq \phi(G(F)). \tag{1.1}$$

If, for every separable field extension K/F, the above equation (1.1) holds, then the norm principle is said to hold for $\phi : G \to T$.

The objective of this paper is to discuss the proof that the norm principle applies to the group of similitudes $\mathrm{GO}(q)$ and the spinor group $\mathrm{Spin}(q)$ for a quadratic form q defined over F.

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2. Preliminaries:

All fields under consideration are assumed to have a characteristic distinct from 2. We denote the set of all non-zero elements of F by $F^\times$, i.e., $F^\times = F \setminus \{0\}$.

Consider a vector space V which is finite dimensional over F. Recall that, by a *bilinear form* b on V, we mean a map $b : V \times V \to F$ satisfying the following:

(1) $b(m_1x + m_2y, z) = m_1b(x, z) + m_2b(y, z)$.
(2) $b(x, m_2y + m_3z) = m_2b(x, y) + m_3b(x, z)$.

for all $x, y, z \in V$ and for $m_1, m_2, m_3 \in F$. A *symmetric* bilinear form is a bilinear form b satisfies $b(x, y) = b(y, x)$ for all vectors x and y in the vector space V. In this paper, by a bilinear form we mean a symmetric bilinear form.

A map $q : V \to F$ is said to be a *quadratic form* on V if

(1) $q(\alpha x) = \alpha^2 q(x)$.
(2) $b_q(x, y) = \frac{1}{2}[q(x + y) - q(x) - q(y)]$ is a (symmetric) bilinear form on V.

We call b_q be the bilinear form associated to the quadratic form q. Also, given a bilinear form b on V, one can associate a quadratic form q_b to b by defining $q_b(x) := b(x, x)$. We refer to (Chapter 1, [5]) for more details. The form $q : V \to F$ is said to be *regular* if $b_q : V \times V \to F$ is nondegenerate. We call (V, q) a quadratic space. Let (V, ϕ) and (W, ψ) be two quadratic spaces over F. We say (V, ϕ) is *isometric* to (W, ψ) (denoted by $(V, \phi) \cong (W, \psi)$) if there is an isomorphism $\sigma : V \to W$ such that $\phi(x) = \psi(\sigma(x))$.

For a quadratic space (V, q) over F, if there exists a non-zero $x \in V$ such that $q(x) = 0$, then we say q is isotropic over F. Otherwise, q is called *anisotropic*. For $n \in \mathbb{N}$, let $n.q$ denotes the n-fold orthogonal sum of q. If $n.q$ is isotropic, then we say q is *weakly isotropic* over F. Note that, a field extension K/F is said to be *totally positive* if any isotropic quadratic form q on K becomes weakly isotropic form on F. For more results on totally positive field extensions, please refer to [4]. A 2-dimensional quadratic form q is said to be *hyperbolic* if $q \cong \langle 1, -1\rangle$. By Witt decomposition theorem (see Chapter 1, Theorem 4.1, [3]), any quadratic form q can be written as $q = q_h \perp q_a$, where q_h denotes the hyperbolic part of q and q_a denotes the anisotropic part of q.

For a quadratic space (V, q) over F, let $D(q)$ denote the set of elements in $F^\times$ represented by q, i.e., $D(q) = \{x \in F^\times |$ there exists $v \in V$such that $q(v) = x\}$.

Lemma 2.1. *Let (V, q) be a quadratic space over F. If $\alpha,\ x \in F^\times$, then $x \in D(q)$ if and only if $\alpha^2 x \in D(q)$.*

Proof. Suppose, $x \in D(q)$, there exists $v \in V$ such that $q(v) = x$. For $\alpha \in F^\times$, $\alpha v \in V$. Since q is a quadratic form, we have $q(\alpha.v) = \alpha^2 q(v) = \alpha^2 x$. Therefore, $\alpha^2 x \in D(q)$. Conversely, if $\alpha^2 x \in D(q)$, then there exists $v \in V$ such that $q(v) = \alpha^2 x$. Hence, $q(\frac{1}{\alpha})v = \frac{1}{\alpha^2} q(v) = x$. Thus $x \in D(q)$. □

Remark 2.2. From the above lemma 2.1, it is easy to see that $D(q)$ consists of a union of cosets of $F^\times/F^{\times 2}$. In general, $D(q)$ need not be a subgroup of $F^\times$. If it forms a subgroup, then we call q a *group form* over F (see Chapter 1, §2, [3]).

However, every element in $D(q)$ has an inverse in $D(q)$. Indeed, for $x \in D(q)$, one can write

$$x^{-1} = (x^{-1})^2. \, x \in D(q).$$

We next discuss construction of Witt rings. Consider a commutative cancellation monoid $(\Sigma, +)$. Define a relation $\sim$ on $\Sigma \times \Sigma$ by

$$(s,t) \ \sim \ (s',t') \text{ if and only if } s + t' = s' + t \in M.$$

Now consider $(\Sigma \times \Sigma)/\sim$. We denote the equivalence class of (s,t) by $[s,t]$. Define an addition by on $(\Sigma \times \Sigma)/\sim$ by

$$[s,t] + [s',t'] := [s+s', t+t']$$

One can easily verify that the addition is well-defined, associative and commutative. For an equivalence class $[s,t]$, we check that $[t,s]$ is the additive inverse for $[s,t]$. Therefore, $(\Sigma \times \Sigma)/\sim$ forms a group under the addition defined. We call the group as *Grothendieck group* and denoted by $\mathrm{Groth}(\Sigma)$. Moreover, if Σ has a multiplication on it, then by defining the product

$$[s,t][s',t'] := [ss' + tt', st' + ts'],$$

$\mathrm{Groth}(\Sigma)$ turns into a ring.

Consider the set $\Sigma(F)$, which consists of all isometry classes of regular F-quadratic forms. The $Groth(\Sigma(F))$ is said to be the *Witt-Grothendieck* ring of the F-quadratic forms and denoted by $\widehat{W}(F) := Groth(\Sigma(F))$. The Witt ring $W(F)$ is obtained by quotienting the ring $\widehat{W}(F)$ by the ideal generated by the hyperbolic spaces $\mathbb{H}$, i.e., $W(F) = \widehat{W}(F)/\langle\mathbb{H}\rangle$ (see [3], Chapter 2 for more details).

Theorem 2.3. *There is an one to one correspondence between the elements of the Witt ring $W(F)$ and the isometry classes of all anisotropic forms over a field F.*

Proof. Any element in $\widehat{W}(F)$ can be written as $[q_1] - [q_2]$, where $[q_1], [q_2]$ are isometry classes of regular quadratic forms of q_1 and q_2 respectively. We claim that any element in $W(F)$ is of the form $[q]$, where q is a regular quadratic form over F. Indeed, $[q_1] - [q_2] = [q_1 \perp (-q_2)] - [q_2 \perp (-q_2)]$. Now, for any scalar $a \in F^{\times}$,

$$\langle a \rangle \perp \langle -a \rangle \cong \langle a, -a \rangle \cong a\langle 1, -1 \rangle = a\mathbb{H} = 0 \in W(F) \tag{2.1}$$

which implies $-\langle a \rangle = \langle -a \rangle \in W(F)$. Since, $[q_2 \perp (-q_2)]$ is a hyperbolic form over F and hence it becomes zero over $W(F)$. Therefore, $[q_1] - [q_2] = [q_1 \perp (-q_2)] \in W(F)$ and hence any element in $W(F)$ is of the form $[q]$ for some quadratic form q. By Witt decomposition theorem, every quadratic form q can be written as $q = q_h \perp q_a$, where q_h denotes the hyperbolic part of q and q_a denotes the anisotropic part of q. Therefore, $[q]$ and $[q_a]$ are equal in $W(F)$. Thus, for each element in $W(F)$, there is some isometry classes of anisotropic form over F. To prove the one to one correspondence, we need to show if $[\phi]$ and $[\psi]$ are equal in $W(F)$ for two anisotropic quadratic forms ϕ and ψ, then ϕ is isometric to ψ. If $[\phi]$ and $[\psi]$ are same in $W(F) = \widehat{W}(F)/\langle\mathbb{H}\rangle$, then $[\phi] = [\psi] \perp \alpha\mathbb{H} \in \widehat{W}(F)$ for some $\alpha \in \mathbb{N} \cup \{0\}$. Hence, $\phi \cong \psi \perp \alpha\mathbb{H}$ as quadratic forms. On the other hand,

ϕ is anisotropic, and so it cannot contain any hyperbolic subpart in it. Therefore $\alpha = 0$ and so, $\phi \cong \psi$. This completes the proof. □

3. Norm principles

Consider a field extension K of F. Then, any F-vector space V can be considered as a K-vector space, given by $V_K := V \otimes_F K$. Thus any F-quadratic space (V, q) can also considered as a K-quadratic space (V_K, q_K), where $q_K = q \otimes_F K$.

3.1. Scharlau's norm principle. Let K/F be a field extension and let s be a non-zero linear functional from K to F. Then for any quadratic space (W, ϕ) over K, we can construct a quadratic space over F by $s_*(W) := (W, s\phi)$. Moreover, if K/F is finite, $\dim_F\, s_*(W) = [K : F] \cdot \dim_K(W)$. We next state the Frobenius Reciprocity theorem.

Theorem 3.1 ([3], Chapter 7, Theorem 1.3)**.** *Let K/F be a field extension and $s : K \to F$ be a non-zero linear functional. Let (V, q) be a quadratic space over F and (W, ϕ) be a quadratic space over K. Then there exists an isometry over F given by*

$$s_*(V_K \otimes_K W) \cong V \otimes_F s_*(W)$$

Consider the rational function field $F(x)$ of F with $[F(x) : F] = n$. Consider an F-basis $\{1, x, \ldots, x^{n-1}\}$ on $F(x)$. The unique non-zero F-linear functional $s : F(x) \to F$ given by $s(1) = 1$ and $s(x) = s(x^2) = \cdots = s(x^{n-1}) = 0$. Then by ([3], Chapter 7, Corollary 2.4),

$$s_*(\langle 1, -x\rangle) = \langle 1, -N_{K/F}(x)\rangle \in W(F). \tag{3.1}$$

Recall that, two F-quadratic forms q_1 and q_2 are called *proportional* if $q_1 \cong \alpha.q_2$ for some $\alpha \in F^\times$ (see [5], Chapter 2, Definition 8.4). In particular, if there exists $\alpha \in F^\times$ such that $q_1 \cong \alpha.q_1$, then α is said to be a *proportionality (similarity) factor* of q_1. For a quadratic form q over a field F, consider the set

$$G(q) := \{\alpha \in F^\times : \alpha.q \cong q\}$$

One can verify that $G(q)$ is a subgroup of $F^\times$. It is called the group of *proportionality (similarity) factors* of q. For any $\alpha \in F^{\times 2}$, by the property of quadratic forms, we have $\alpha.q$ is isometric to q. Thus, α is a similarity factor for q. Therefore, $F^{\times 2} \subseteq G(q)$. We now discuss Scharlau's norm principle.

Theorem 3.2 (Scharlau)**.** *Let K be a finite field extension of F and q be a regular quadratic form over F. Then for any $x \in K^\times$, the following inclusion holds*

$$x \in G(q_K) \implies N_{K/F}(x) \in G(q).$$

In other words,

$$N_{K/F}(G(q_K)) \subseteq G(q)$$

Proof. Let $x \in K^\times$. Consider the intermediate field $F(x)$ with $F \subseteq F(x) \subseteq K$. For convenience, let us denote the field $F(x)$ by E.

Case (i): Suppose K/E is an even degree extension, say, $[K : E] = 2m$. Then $N_{K/E}(x) = x^{2m}$. By the multiplicative property of norm, we have

$$N_{K/F}(x) = N_{E/F} \cdot (N_{K/E}(x)) = N_{E/F} \cdot (x^{2m}) = (N_{E/F}(x^m))^2 \in F^{\times 2}$$

Therefore, $N_{K/F}(x) \in G(q)$.

Case (ii): Suppose, $[K : E] = 2m+1$, an odd degree extension. As $x \in G(q_K)$, we have $x.\ q_K \cong q_k$, i.e., $\langle 1, -x\rangle \otimes_K q_K = 0 \in W(K)$. By ([3], Chapter 7, Theorem 2.5), the map $W(E) \to W(K)$ is injective. Hence $\langle 1, -x\rangle \otimes_E q_E = 0 \in W(E))$ and so $x \in G(q_E)$. Now,

$$\begin{aligned} N_{K/F}(x) =& N_{E/F} \cdot (N_{K/E}(x)) \\ =& N_{E/F} \cdot (x^{2m+1}) \\ =& N_{E/F}(x) \cdot (N_{E/F}(x^m))^2 \end{aligned}$$

Thus if we show that $N_{E/F}(x) \in G(q)$, we are done. So we can assume $K = F(x)$ and let $s : K \to F$ be the unique F-linear functional defined as earlier this section. Applying the transfer s_* to the equation $\langle 1, -x\rangle \otimes_K \ q_K = 0 \in W(K)$, we get

$$\begin{aligned} 0 &= s_*(q_K \otimes_K \langle 1, -x\rangle_K) \\ &\cong q \otimes_F s_*(\langle 1, -x\rangle_K) \text{ (by Theorem 3.1)} \\ &\cong q \ \otimes_F \langle 1, -N_{K/F}(x)\rangle \ \in W(F) \text{ (by Equation 3.1)} \end{aligned}$$

Hence $\langle 1, -N_{K/F}(x)\rangle \otimes_F q = 0 \in W(F)$ and so $N_{K/F}(x) \in G(q)$. □

Remark 3.3. Let $\mathrm{GO}(q)$ be the group of similitudes and $\phi : \mathrm{GO}(q) \to \mathbb{G}_m$ be the multiplier homomorphism (see [2], Chapter 3 for more details). The norm principle for ϕ readily follows from the above Theorem 3.2 (see [1], example 3.3).

3.2. Knebusch's norm principle. Recall the notion of $D(q)$ for a F-quadratic space (V, q) defined in section 2 as follows

$$D(q) = \{d \in F^\times |\ q(v) = d \text{ for some } v \in V\}.$$

In the earlier section 3.2, we discuss the group of similarity factors $G(q)$ with respect to the norm map for a finite field extension. It will be of interest to establish a parallel result for $D(q)$ also. However, the main issue is that the set $D(q)$ may not form a subgroup of the multiplicative group $F^\times$. For example, the quadratic form $q = \langle -1\rangle$ over $\mathbb{R}$ does not represent 1. In fact, $D(q)$ need not be closed under multiplication. For example, as discussed in ([3], Chapter 1, §2), consider the quadratic form $q = \langle 1, 1, 1\rangle = x^2 + y^2 + z^2$ over $\mathbb{Q}$. Then clearly, $1, 2, 2^{-1}, 14 \in D(q)$. But the product $2^{-1}.14 = 7 \notin D(q)$ as a sum of three squares over $\mathbb{Q}$ cannot be used to express 7 (see Legendre's three-square theorem).

Theorem 3.4 (Knebusch)**.** *Let K be a field extension of F and $[K : F] = n$. Let q be a regular F-quadratic form. Suppose $x \in K^\times$. If x is represented by the form $q \otimes_K K$, then the norm of x, $N_{K/F}(x) \in D^n(q)$.*

Proof. If q is isotropic over F, then by ([3], Chapter 1, Theorem 3.4), $D(q) = F^\times$. Hence, $N_{K/F}(x) \in F^\times = D(q)$. Let's suppose that q is an anisotropic F-quadratic form. For $m \geq 1$, let $D^m(q)$ represent the collection of products of m elements in $D(q)$. Let $d \in D(q)$. Then for any $c \in F^\times$,

$$c^2 = d \cdot ((c/d)^2 \cdot d) \in D(q) \cdot D(q) = D^2(q).$$

Thus, $F^{\times 2} \subseteq D^2(q)$. More generally, $F^{\times 2m} \subseteq D^{2m}(q)$.

We now use induction on $[K : F] = n$ to prove the theorem. If $n = 1$, then $K = F$ and we are done. Assume $n \geq 2$ in the following.

Case (i): Let $x \in F$. If $[K : F] = 2m$, then

$$N_{K/F}(x) = x^{2m} \in F^{\times 2m} \subseteq D^{2m}(q).$$

If $[K : F] = 2m + 1$, then $N_{K/F}(x) = x^{2m+1} \in x \cdot F^{\times 2m}$. Since K/F is an odd degree extension, by ([3], Chapter 7, Corollary 2.9), $x \in D(q_K) \implies x \in D(q)$. Hence

$$N_{K/F}(x) \in x \cdot F^{\times 2m} \subseteq D(q) \cdot D^{2m}(q) = D^n(q)$$

Case (ii): Suppose $x \notin F$. Let $E = F(x)$ and consider $F \subseteq E \subseteq K$ with $[K : E] = m,\ [E : F] = m',\ n = mm'$.

Suppose $E \subsetneq K$. Then $m > 1$ and by the above case i, $N_{K/E}(x) \in D^m(q_E)$ Therefore, $N_{E/F} \cdot N_{K/E}(x) \in N_{E/F}(D^m(q_E))$ Hence,

$$N_{K/F}(x) \in N_{E/F}(D^m(q_E)) \tag{3.2}$$

As $m > 1$ and so $m' < n$, by induction hypothesis on E/F, we have

$$N_{E/F}(x) \in D^{m'}(q),\ \text{for each } x \in D(q_E).$$

Hence, using 3.2, $N_{K/F}(x) \in D^{mm'}(q) = D^n(q)$.

Suppose $E = K$. Let $p(t)$ be the minimal polynomial of x over F, so

$$K = F(x) \cong F[t]/(p(t))$$

As $D(q_K)$ is closed under inverses, and $x \in D(q_K)$, we have $x^{-1} \in D(q_K)$. So there exists $f_1, \dots, f_d \in F[t]$, satisfying

$$q(f_1(x), \dots, f_d(x)) = x^{-1}.$$

Given that the minimal polynomial of x over F is $p(t)$. So we have

$$t \cdot q(f_1(t), \dots, f_d(t)) = 1 + p(t)h(t) \tag{3.3}$$

where $d = \dim q,\ h(t), f_i(t) \in F[t]$ with $r := \max\{\deg(f_i)\} \leq n - 1$. Since, q is anisotropic, 3.3 shows that $n_0 := \deg(h) = 2r + 1 - n \leq 2(n-1) + 1 - n = n - 1$.

If $h(t) = c \cdot h_1(t) \cdots h_s(t)$, where $c \in F^\times$ and the h_i's are monic irreducible polynomials in $F[t]$, then c is the leading co-efficient of $1 + p(t)h(t)$. Using 3.3, we have $c \in D(q)$.

Now if $s = 0$ in the decomposition of h, i.e, if $h(t)$ is a constant polynomial c, then $n_0 = 0$ and

$$N_{K/F}(x) = (-1)^n p(0) = (-1)^{n+1} h(0)^{-1} = (-1)^{n+1} c^{-1} = (-1)^{2r+1+1} c^{-1} = c^{-1}$$

Since $c \in D(q)$, we have $N_{K/F}(x) \in D(q)$. Suppose $s \geq 1$ in the decomposition of h, and x_i is a zero of h_i in an algebraic closure of F. Then from (3.3), we have

$$x_i^{-1} = q(f_1(x_i), \ldots, f_d(x_i)) \in D(q_{F(x_i)}).$$

Since $[F(x_i) : F] \leq \deg\, h \leq n-1$, by induction hypothesis on $F(x_i)/F$,

$$N_{F(x_i)/F}(x_i) = (-1)^{deg\ h_i} h_i(0) \in D^{deg\ h_i}(q).$$

Taking the product of these over i, we get

$$(-1)^{n_0} h_1(0) \cdots h_s(0) = (-1)^{n_0} c^{-1} h(0) \in D^{n_0}(q).$$

Since $c^{-1} \in D(q)$, we have

$$(-1)^{n_0} h(0) \in D^{n_0+1}(q).$$

As $n_0 + 1 \equiv n \ (mod\ 2)$ and recalling that $F^{\times 2m} \subseteq D^{\times 2m}(q)$, we have

$$(-1)^{n+1} h(0) \in D^n(q).$$

Therefore, $N_{K/F}(x) = (-1)^n p(0) = (-1)^{n+1} h(0)^{-1} \in D^n(q)$. The proof is now complete. □

Remark 3.5. Consider a quadratic space (V, q) over F, and let $\Gamma^+(V, q)$ be the even Clifford group of (V, q). Consider the spinor norm homomorphism $\mathrm{Sn} : \Gamma^+(V, q) \to \mathbb{G}_m$. The elements in the $\mathrm{Im}(\mathrm{Sn}_F)$ is the product of elements in $D(q)$. The kernel of the spinor norm homomorphism is called the spinor group of (V, q) and denoted by $\mathrm{Spin}(V, q)$. The norm principle for $\mathrm{Spin}(V, q)$ follows the above Theorem 3.4 (see [1], Example 3.2).

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