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A NEW APPROACH ON GRAPH TOPOLOGICAL SPACES

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Abstract. Many topological versions are introduced in graph theory by different authors. We attempt to introduce a new graph topological space with examples in this work. Moreover, we look into the new topology's characteristics.

1. Introduction

Topology and Graph Theory are two significant and closely associated fields of mathematics. As a mathematical tool, graph theory is widely used in various fields namely operations research, chemistry, sociology, genetics and so on. It is also used in the resolution of numerous challenging real-world situations. Topology is an important area of mathematics that greatly influences other areas of the subject. Topological structures are mathematical representations that are used in analysis of data irrespective of dimension.

The study of graph theory by means of topology is a promising area of study. A large number of studies focus on topologizing graphs by constructing topologies from graphs using numerous techniques. A topology on a directed graph was established in 1968 by T.N. Bhargava and T. J. Ahlborn. In 2013, Saba Nazar Faisel Al-khafaj and Abedal-Hamza Mahdi Hamzi created a topology on an undirected graph. M. Shorky created topology on graphs using operations on graphs in 2015 [7]. In 2018, we saw the creation of topologies on the edge set of the graphs by K.A. Abdu and A. Kilicman [6]. Hatice Kubra Sari and Abdullah Kopuzlu examined the topological spaces produced by undirected graphs in 2020 [3].

All earlier studies of the topologies connected to graphs were vertex- or edge-based. A topology for graphs in terms of spanning subgraphs was introduced by Karunakaran K. and G. Suresh Singh in 2007 [5]. This motivated us to define an edge induced subgraph topology. We investigate certain generalized topological structure by which we can relate graph theory and topology. In this work we presented some preliminary character of the topological structure. Let E be the edge set of a graph G and consider the collection of edge induced subgraphs generated by the subsets of E . We define the topology $\mathcal{T}$ of edge induced subgraphs generated by the subsets of E. We follow [2] for all terminologies and notations that are not provided here.

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2. Preliminaries

In this section we present some preliminaries of graph theory and topology.

Definition 2.1. Let $G = (V, E)$: V is a set of elements (V may be empty) and E is a set of unordered pairs of elements of V. The elements of V are called **vertices** and that of E are called **edges**. The set V is called the **vertex set** of G and E is called **edge set** of G. Then the ordered pair of disjoint sets (V, E) is known as a **graph**, and simply denoted as G.

Definition 2.2. If the set of edges and vertices are finite, the graph G is **finite**, otherwise it is **infinite**. If the edge set, E is empty the graph is called a **vertex graph**.

Definition 2.3. If both vertex set and edge set are empty, the graph is called **null Graph** (or **void graph** or **empty graph**). We denote the null graph by ϕ.

Definition 2.4. A graph H is a subgraph of a graph G if all the vertices and edges of H are in G. Then $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, where $V(H)$, $(V(G))$ and $E(H)$, $(E(G))$ are the vertex set of H (vertex set of G) and edge set of H (edge set of G) respectively.

Definition 2.5. Let G be a graph and M a nonempty subset of $E(G)$. A subgraph of G whose edge set is M and the vertex set is the set of all vertices incident to the edges in M, is said to be the **edge induced subgraph** of G induced by M, and it is denoted by $\langle M \rangle$.

Definition 2.6. Let $G = (V, E)$ be a graph and let $H = (V_H, E_H)$ be an edge induced subgraph of G. The **complement of the edge induced subgraph of H** with respect to G is the gaph $H' = (V', E')$; where $E' = E \setminus E_H$ and the vertex set V' is the set of all vertices incident to the edges in $E(H')$.

i.e., the complement of the edge induced subgraph of H with respect to G is the edge induced subgraph generated by E'.

Definition 2.7. Let $G = (V, E)$ be a graph and let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be edge induced subgraphs of G.

The **union** of H_1 and H_2 is the edge induced subgraph $H_1 \cup H_2$ with edge set $E_1 \cup E_2$ and the vertex set is the set of all vertices incident to the edges in $E_1 \cup E_2$.

The **intersection** of H_1 and H_2 is the edge induced subgraph $H_1 \cap H_2$ with edge set $E_1 \cap E_2$ and the vertex set is the set of all vertices incident to the edges in $E_1 \cap E_2$.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be **identical** if $V_1 = V_2$ and $E_1 = E_2$.

Definition 2.8. Let X be any non-empty set and τ be a family of subsets of X satisfying the following conditions:

- **a)** The set X and the empty set ϕ belong to τ.
- **b)** Arbitrary union of members of τ is in τ.
- **c)** Finite intersection of members of τ is in τ.

Then τ is called a **topology** for X and the members of τ are called **open sets**. The ordered pair (X, τ) is called a topological space.

Definition 2.9. A subset C of a topological space X is **closed** provided that its complement $X \backslash C$ is an open set.

3. Edge Induced Subgraph Topology

In this paper a null graph is denoted by ϕ, whose vertex set and edge set are empty. Now, we define an edge induced subgraph topology of a given graph as follows,

Definition 3.1. Let $G = (V, E)$ be a graph of size greater than or equal to 1 and let $\mathcal{T}$ be a collection of edge induced subgraphs of G, generated by the subsets of E satisfying the following properties:

- **a)** The graph G and the empty graph ϕ belong to $\mathcal{T}$.
- **b)** Arbitrary union of members of $\mathcal{T}$ is in $\mathcal{T}$.
- **c)** Finite intersection of members of $\mathcal{T}$ is in $\mathcal{T}$.

Then $\mathcal{T}$ is called an **edge induced subgraph topology** on the graph G and the members of $\mathcal{T}$ are called **edge induced open subgraphs** in G. The ordered pair $(G, \mathcal{T})$ is called an **edge induced subgraph topological space**.

Remark 3.2. Let $G = (V, E)$ be a graph. Then the collection of all edge induced subgraphs of G is the largest edge induced subgraph topology on the graph G and it is called the **discrete edge induced subgraph topology** on G.

Remark 3.3. Let $G = (V, E)$ be a graph. Then the collection $\Big\{\phi, G\Big\}$ is an edge induced subgraph topology on G. It is the smallest edge induced subgraph topology associated with the graph G and it is called the **indiscrete edge induced subgraph topology** on G.

Definition 3.4. Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. An edge induced subgraph H of G is said to be **an edge induced closed subgraph** if the complement of the edge induced subgraph of H is open in G.

Proposition 3.5. *Let G be a graph and $\mathcal{T}_1$, $\mathcal{T}_2$ be any two edge induced subgraph topologies on G . Then $\mathcal{T}_1 \cap \mathcal{T}_2$ is an edge induced subgraph topology on G .*

Proof. Given that $\mathcal{T}_1$ and $\mathcal{T}_2$ are any two edge induced subgraph topologies.

$$\text{Set } \mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2.$$

Obviously empty graph and G belong to $\mathcal{T}$. Let H_1, $H_2 \in \mathcal{T}$, which implies H_1, H_2 belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$. It follows that $H_1 \cap H_2$ belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$ and hence in $\mathcal{T}$.

Let $\Big\{H_\alpha, \alpha \in A\Big\}$ be an arbitrary collection of edge induced subgraphs in $\mathcal{T}$, then $\bigcup\limits_{\alpha \in A} H_\alpha$ belong to both $\mathcal{T}_1$ and $\mathcal{T}_2$ and hence $\bigcup\limits_{\alpha \in A} H_\alpha$ belongs to $\mathcal{T}$. □

Remark 3.6. Union of two edge induced subgraph topologies need not be an edge induced subgraph topology.

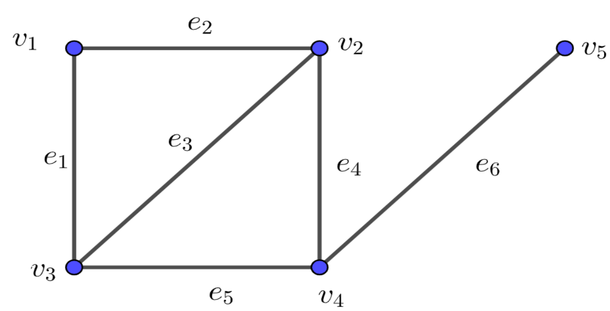

FIGURE 1. G

Illustration 1. *Consider the graph in Figure 1. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two edge induced subgraph topologies such that* $\mathcal{T}_1 = \Big\{\phi, \langle e_1\rangle, G\Big\}$ *and* $\mathcal{T}_2 = \Big\{\phi, \langle e_6\rangle, G\Big\}$*. This gives* $\mathcal{T}_1 \cup \mathcal{T}_2 = \Big\{\phi, \langle e_1\rangle, \langle e_6\rangle, G\Big\}$*, but it is not an edge induced subgraph topology .*

Definition 3.7. Let $\mathcal{C}$ be an arbitrary collection of edge induced graph topological spaces. Let G_1, $G_2 \in \mathcal{C}$ and define a relation $\sim$ on $\mathcal{C}$ as $G_1 \sim G_2$ if and only if G_1 and G_2 are isomorphic and the edge induced subgraph topology on G_1 is same as that on G_2.

Theorem 3.8. *The above relation '$\sim$' is an equivalence relation on $\mathcal{C}$.*

Proof. **i)** Let $\mathcal{T}_1$ be the edge induced subgraph topology on G_1. Then obviously $G_1 \sim G_1$. Hence $\sim$ is reflexive.

ii) Since $G_1 \sim G_2$, we have G_1 and G_2 are isomorphic and edge induced subgraph topology on G_1 is same as that on G_2. This implies that $G_2 \sim G_1$. Thus $\sim$ is symmetric.

iii) Since $G_1 \sim G_2$ and $G_2 \sim G_3$, then G_1 is isomorphic to G_2 and G_2 is isomorphic to G_3, hence G_1 is isomorphic to G_3 and also topology on G_1 is same as that on G_2 and topology on G_2 is same as that on G_3, hence the topology on G_1 is same as that on G_3. It follows that $G_1 \sim G_3$. Therefore $\sim$ is transitive.

Thus $\sim$ is an equivalence relation on $\mathcal{C}$.

□

Theorem 3.9. *The edge induced closed subgraphs of a topological space $(G, \mathcal{T})$ have the following properties:*

a) *The graph G and the empty graph ϕ are edge induced closed subgraphs.*
b) *Arbitrary intersection of edge induced closed subgraphs is an edge induced closed subgraph.*
c) *Finite union of edge induced closed subgraphs is an edge induced closed subgraph.*

Proof. **a)** The complement edge induced subgraphs of G and ϕ are ϕ and G. Moreover, both are edge induced open subgraphs and so G and ϕ are edge induced closed subgraphs.

b) Consider a family $\Big\{H_\alpha, \alpha \in A\Big\}$ of edge induced closed subgraphs of G. Now complement edge induced subgraph of $\Big(\bigcap_{\alpha\in A} H_\alpha\Big)$ is $\Big(\bigcup_{\alpha\in A} H'_\alpha\Big)$ which is the union of edge induced open subgraphs and so is open.

c) Consider a family $\Big\{H_k : k = 1, 2, \ldots, n\Big\}$ of edge induced closed subgraphs of G. Now complement edge induced subgraph of $\Big(\bigcup_{k=1}^{n} H_k\Big)$ is $\Big(\bigcap_{k=1}^{n} H'_k\Big)$ which is the intersection of edge induced open subgraphs and so is open. □

Definition 3.10. Let $(G, \mathcal{T})$ be an edge induced subgraph topological space. A subfamily $\mathcal{G}$ of $\mathcal{T}$ is a base of $\mathcal{T}$ if every member of $\mathcal{T}$ is the union of some members of $\mathcal{G}$.

Suppose $(G, \mathcal{T})$ is an discrete edge induced subgraph topological space. Then the collection $\mathcal{G}$, of all single edge subgraphs of G is a base for $\mathcal{T}$. The next theorem gives necessary and sufficient conditions that a family of edge induced subgraphs is a base for edge induced subgraph topology.

Theorem 3.11. *Let $(G, \mathcal{T})$ be an edge induced subgraph topological space and let $\mathcal{G} \subset \mathcal{T}$. Then $\mathcal{G}$ is a base for the topological space if and only if*

i) $\bigcup \mathcal{G}$ *is that of G.*

ii) *For each $e \in E(G), \exists\ H_i \in \mathcal{G} : e \in E(H_i) \subset E(G)$.*

Proof. Given that $(G, \mathcal{T})$ be an edge induced subgraph topological space and $\mathcal{G} \subset \mathcal{T}$. Suppose $\mathcal{G}$ is a base for the topological space. First part is obvious.
Let $e \in E(G)$ and let K be an edge induced open subgraph of G containing the edge e. By the definition of base, K can be expressed as the union of some members of $\mathcal{G}$. *i.e.*,

$$K = \bigcup_{i\in I} H_i \text{ , where each } H_i \in \mathcal{G}.$$

Since $e \in E(K)$, we have $e \in E(H_i) \subset E(K) \subset E(G)$ for some i.

Conversely suppose that the two conditions are satisfied. We have to prove that the collection $\mathcal{G}$, of edge induced open subgraphs of G is a base for the topological space. It is enough, if we show that the every element of $\mathcal{T}$ is the union of members of $\mathcal{G}$. Consider an edge induced open subgraph, K of G.
By the assumption, for each $e \in E(G), \exists\ H_i \in \mathcal{G}$ such that $e \in E(H_i) \subset E(G)$.
Hence for each $e \in E(K), \exists\ H_i \in \mathcal{G} : e \in E(H_i) \subset E(G)$.
Therefore, we have

$$K = \bigcup_{e\in E(K)} H_i.$$

That is, K can be expressed as the union of members of $\mathcal{G}$. Hence, $\mathcal{G}$ is a base for the edge induced graph topological space $(G, \mathcal{T})$. □

4. Edge Induced Neighbourhood of an Edge and Edge Induced Interior Subgraph

Definition 4.1. Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let K be an edge induced subgraph of G. For an edge $e \in E(K)$, K is an **edge induced neighbourhood of** e, $\exists\, H_i \in \mathcal{T}$ provided $e \in E(H_i)$ and $H_i \subset K$.
Now e is called **interior edge** of K. The collection of all interior edges of K is denoted by K'.

Definition 4.2. An **edge induced interior subgraph of K** is the subgraph generated by K'. It is denoted by **Edg Int K**. *i.e.*,

$$\textbf{Edg Int K} = \langle K' \rangle.$$

Theorem 4.3. *Let $(G, \mathcal{T})$ be a graph topological space and let H be an edge induced subgraph of G. Then*

1) ***Edg Int H*** *is open in G.*
2) ***Edg Int H*** *is the largest edge induced open subgraph with* $E($***Edg Int H***$) \subset E(H)$.

Proof. Given that $(G, \mathcal{T})$ be a graph topological space and let H be an edge induced subgraph of G.

1) Consider an edge induced open subgraph K of G such that $K \subset H$. Then every edge of K is an interior edge of H.
   It follows that every edge induced open subgraph K, with $E(K) \subset E(H)$ is a subgraph of **Edg Int H**.
   Hence **Edg Int H** is the union of all such edge induced open subgraphs of G.
   i.e., **Edg Int H** is an edge induced open subgraph of G.
2) From (1) **Edg Int H** is open in G and from the definition, $E($**Edg Int H**$) \subset E(H)$. Let K_1 be any open subgraph of G with $E(K_1) \subset E(H)$. We have to show that $E(K_1) \subset E($**Edg Int H**$)$. For $e \in E(K_1)$, $e \in E(H)$ and hence e is an interior edge of H.

   i.e., $e \in E($**Edg Int H**$)$. Thus $E(K_1) \subset E($**Edg Int H**$)$.

□

Theorem 4.4. *Let $G = (V, E)$ be a graph and $\mathcal{T}$ be an edge induced subgraph topology on G. If H and K are any two edge induced subgraphs of G, then*

a) *The edge induced subgraph H, is open if and only if H and* ***Edg Int H*** *are identical.*
b) *If $H \subseteq K$ then* ***Edg Int H*** $\subseteq$ ***Edg Int K***.
c) ***Edg Int (H*** $\cap$ ***K)*** *and* ***Edg Int H*** $\cap$ ***Edg Int K*** *are identical.*
d) ***Edg Int H*** $\cup$ ***Edg Int K*** $\subset$ ***Edg Int (H*** $\cup$ ***K)***.

Proof. **a)** Assume that an edge induced subgraph H of G is open. By Theorem 4.3 (2), **Edg Int H** is the largest edge induced open subgraph with, $E($**Edg Int H**$) \subset E(H)$. Then $H \subset$ **Edg Int H**.

Which follows, $H =$ **Edg Int H** .

Conversely, let $H =$ **Edg Int H** . By Theorem 4.3.(1), **Edg Int H** is open in G, hence H is open in G.

b) We have $H \subseteq K$. Let $e \in E($**Edg Int H**), then e is an interior edge of H. By definition $\exists\ H_i \in \mathcal{T}$ provided $e \in E(H_i)$ and $H_i \subset H$. *i.e.*, $\exists\ H_i \in \mathcal{T}$ provided $e \in E(H_i)$ and $H_i \subset K$, since $H \subseteq K$.
Thus e is an interior edge of K, it follows that, $e \in E($**Edg Int K**).
i.e., , **Edg Int H** $\subseteq$ **Edg Int K** .

c) Suppose H and K be any two edge induced subgraphs of G. Now we have E(H ∩ K) = E(H) ∩ E(K). Also, (H ∩ K) ⊆ H. and (H ∩ K) ⊆ K. Then by the part (b),

$$\textbf{Edg Int (H} \cap \textbf{K)} \subseteq \textbf{Edg Int H},$$
$$\textbf{Edg Int (H} \cap \textbf{K)} \subseteq \textbf{Edg Int K}$$
$$\Longrightarrow \text{E}(\textbf{Edg Int (H} \cap \textbf{K)}) \subseteq \text{E}(\textbf{Edg Int H}).$$
$$\text{E}(\textbf{Edg Int (H} \cap \textbf{K)}) \subseteq \text{E}(\textbf{Edg Int K}).$$

So we get,
E(**Edg Int (H ∩K)**) ⊆ E(**Edg Int H**) ∩ E(**Edg Int K**).
Edg Int (H ∩ K) ⊆ **Edg Int H** ∩ **Edg Int K** .
By Theorem 4.3(1), **Edg Int H** is open in G such that
E(**Edg IntH**) $\subseteq E(H)$ and **Edg Int K** is open in G such that
E(**Edg IntK**) $\subseteq E(K)$. It gives,
E(**Edg Int H**) ∩ E(**Edg Int K**) ⊆ E(H) ∩ E(K)= E(H ∩ K).
It follows, **Edg Int H** ∩ **Edg Int K** is an edge induced open subgraph with this property. But **Edg Int (H ∩ K)** is the largest open subgraph with E(**Edg Int (H ∩ K)**) $\subseteq E(H \cap K)$.
Which means, **Edg Int H** ∩ **Edg Int K** ⊆ **Edg Int (H ∩ K)**.
Edg Int (H ∩ K) and **Edg Int H** ∩ **Edg Int K** are identical graphs.

d) From part (2) of Theorem 4.3, we get **Edg Int H** is open in G such that, E(**Edg IntH**) $\subseteq E(H)$. Also **Edg Int K** is open in G such that
E(**Edg IntK**) $\subseteq E(K)$. Which implies,
E(**Edg Int H**) ∪ E(**Edg Int K**) ⊆ E(H) ∪ E(K)= E(H ∪ K) . That is, **Edg Int H** ∪ **Edg Int K** is an edge induced open subgraph with this property. But **Edg Int (H ∪ K)** is the largest open subgraph with E(**Edg Int (H ∪ K)**) ⊆ E(H ∪ K).
Therefore, **Edg Int H** ∪ **Edg Int K** ⊆ **Edg Int (H ∪ K)**.

□

Remark 4.5. **Edg Int H** ∪ **Edg Int K** and **Edg Int (H ∪ K)** are not identical.

Illustration 2. *Consider the following graph, G in Figure 2 and $\mathcal{T}$ is the edge induced subgraph topology on G.*

$$\mathcal{T} = \Big\{\phi, \langle e_1\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle, G\Big\}$$

*Let $H = \langle e_1, e_2, e_3\rangle$ and $K = \langle e_4, e_5, e_6\rangle$. Then **Edg Int H** $= \langle e_1\rangle$ and*
***Edg Int K** $= \phi$. **Edg Int H** ∪ **Edg Int K** $= \langle e_1\rangle$.*

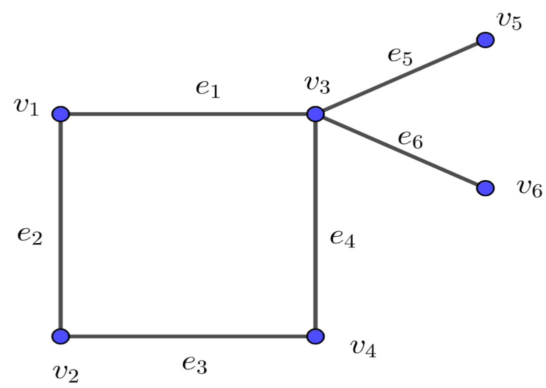

FIGURE 2. G

*But $H \cup K = \langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle$ and **Edg Int (H ∪ K)** $= G$. Therefore **Edg Int H** ∪ **Edg Int K** and **Edg Int (H ∪ K)** are not identical.*

5. Limit Edges of an Edge Induced Subgraph and Edge Induced Closure Subgraph

Definition 5.1. Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let H be an edge induced subgraph of G. An edge $e \in E(G) \backslash E(H)$ is an **limit edge** of H if for all edge induced open subgraph K' in $\mathcal{T}$ with $e \in E(K')$, $E(H) \cap E(K')$ is a non-empty edge set.

Notation 1. *The collection of all limit edges of an edge induced subgraph H, is denoted by H^*.*

Definition 5.2. Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let H be an edge induced subgraph of G. **edge induced closure subgraph** of H is defined as edge induced subgraph generated by $E(H) \cup H^*$ and it is denoted by **Edg Cl(H)**. That is, **Edg Cl(H)** $= \langle E(H) \cup H^* \rangle$.

Lemma 5.3. *Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. If H is an edge induced subgraph of G, then H is an edge induced closed subgraph if and only if **Edg Cl(H)** and H are identical.*

Proof. Let H be an edge induced closed subgraph of G. Then the complement of edge induced subgraph of H, denoted as K is an edge induced open subgraph of G. By definition,

$$\textbf{Edg Cl(H)} = \langle E(H) \cup H^* \rangle.$$

Hence, E(H) ⊆ E(**Edg Cl(H)**). Therefore, $H \subseteq$ **Edg Cl(H)** .
Now choose an edge $e \in E($**Edg Cl(H)**$)$, then $e \in E(H)$ or $e \in H^*$.
If $e \in E(H)$, then **Edg Cl(H)** $\subseteq H$.
So if e is a limit edge of H, we have to show that $e \in E(H)$.

If possible, $e \notin E(H)$. Then, $e \in E(G) \setminus$ E(H) and $\langle E(G) \setminus E(H)\rangle = K_1$ is an edge induced open subgraph of G.
It gives that e is an interior edge of K_1, hence $\exists$ an $H' \in \mathcal{T}$ such that $e \in H'$ and $H' \subseteq K_1$. This implies, $E(H') \cap$ E(H) $= \phi$, which contradicts that e is a limit edge of H.
Thus, **Edg Cl(H)** $\subseteq H$. This gives, **Edg Cl(H)** and H are identical.

Conversely suppose that **Edg Cl(H)** and H are identical. It is sufficient to show that H is an edge induced closed subgraph of G. For it is enough to show that $\langle E(G) \setminus E(H)\rangle = K$ is an edge induced open subgraph of G. Consider an edge, then $e \in E(K)$, but $e \notin E(H) \implies e \notin E(\textbf{Edg Cl(H)})$. It follows that e is not a limit edge of H. Therefore there exists $H' \in \mathcal{T}$ such that $e \in E(H')$ and $E(H') \cap E(H) = \phi$. That is, $e \in E(H')$ and $H' \subseteq K$.
Which implies e is an interior edge of K. Since e is arbitrary, every edge of K is an interior edge of K. Hence, K is an edge induced open subgraph of G. □

Proposition 5.4. *Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let H be an edge induced subgraph of G. Then H is an edge induced closed subgraph of G, if and only if every limit edge of H is in $E(H)$.*

Proof. Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let H be an edge induced closed subgraph of G and e be a limit edge of H. We have to show that $e \in E(H)$. If possible, $e \notin E(H)$. Then, then $e \in E(G) \setminus E(H)$.
Since H is closed, $\langle E(G) \setminus E(H)\rangle = K'$ is open in G. By definition, $\exists\ H^* \in \mathcal{T}$ such that $e \in H^*$ and $H^* \subseteq K'$. That means, $E(H^*) \cap E(H) = \phi$, which is a contradiction. Since e is arbitrary every limit edge of H is in $E(H)$.

Conversely assume that every limit edge of H is in $E(H)$. If H^* is the collection of all limit edges of H, then $H^* \subseteq E(H)$. So we get $E(H) \cup H^* = E(H)$.
Therefore, $\langle E(H) \cup H^*\rangle = \textbf{Edg Cl(H)} \implies \textbf{Edg Cl(H)} = H$.
By Lemma 5.3, H is an edge induced closed subgraph of G. □

Theorem 5.5. *Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be an edge induced subgraph topology on G. Let H and K be an any two edge induced subgraphs of G. Then*

- **a)** ***Edg Cl(H)*** *is the smallest edge induced closed subgraph such that $E(H) \subseteq E(\textbf{Edg Cl(H)})$.*
- **b)** *If $H \subseteq K$ then,* ***Edg Cl(H)*** $\subseteq$ ***Edg Cl(K)***.
- **c)** ***Edg Cl(H*** $\cup$ ***K)*** *and* ***Edg Cl(H)*** $\cup$ ***Edg Cl(K)*** *are identical.*
- **d)** ***Edg Cl(H*** $\cap$ ***K)*** $\subseteq$ ***Edg Cl(H)*** $\cap$ ***Edg Cl(K)***.

Proof. **a)** By definition, **Edg Cl(H)** $= \langle E(H) \cup H^*\rangle$.
Then $E(H) \subseteq$ **E(Edg Cl(H))**. We have to show **Edg Cl(H)** is closed in G. If e is a limit edge of **Edg Cl(H)**, there exists $H' \in \mathcal{T}$ such that,

$$e \in E(H') \text{ and } E(H') \cap \textbf{E(Edg Cl(H))} \neq \phi.$$

That is, $E(H') \cap$ E(H) $\neq \phi$. Which implies, e is a limit edge of H. Hence $e \in E(\textbf{Edg Cl(H)})$.

By Proposition 5.4, **Edg Cl(H)** is an edge induced closed subgraph of G. If K is an edge induced closed subgraph such that $E(H) \subseteq E(K)$. We

have to show that **Edg Cl(H)** $\subseteq$ K. By Lemma 5.3,
Edg Cl(H) = $\langle$ E(H) $\cup$ H^* $\rangle$ = H $\subseteq$ K , since E(H)$\subseteq$ E(K).

b) If $H \subseteq K$, then $E(H) \subseteq E(K)$ and $H^* \subseteq K^*$, where H^* and K^* are the collection of limit edges of H and K respectively. By definition,
Edg Cl(H) $= \langle E(H) \cup H^* \rangle \subseteq \langle E(K) \cup K^* \rangle =$ **Edg Cl(K)**.

c) Since $H \subseteq H \cup K$, $K \subseteq H \cup K$, By part (b),
Edg Cl(H) $\subseteq$ **Edg Cl(H $\cup$ K)** and **Edg Cl(K)** $\subseteq$ **Edg Cl(H $\cup$ K)**
Edg Cl(H) $\cup$ **Edg Cl(H)** $\subseteq$ **Edg Cl(H $\cup$ K)**.

By part (a), **Edg Cl(H)** is the smallest edge induced closed subgraph such that $E(H) \subseteq E($**Edg Cl(H)**$)$ and **Edg Cl(K)** is the smallest edge induced closed subgraph such that $E(H) \subseteq E($**Edg Cl(K)**$)$. Hence, **Edg Cl(H)** $\cup$ **Edg Cl(K)** is an edge induced closed subgraph such that $E(H) \cup E(K) \subseteq E($**Edg Cl(H)** $\cup$ **Edg Cl(K)**$)$. That is,
Edg Cl(H) $\cup$ **Edg Cl(K)** is an edge induced closed subgraph such that $E(H \cup K) \subseteq E($**Edg Cl(H $\cup$ K)**$)$.

By definition, **Edg Cl(H $\cup$ K)** is the smallest edge induced closed subgraph with E(H $\cup$ K) $\subseteq$ E(**Edg Cl(H $\cup$ K)**). Hence we have,
Edg Cl(H $\cup$ K) $\subseteq$ **Edg Cl(H)** $\cup$ **Edg Cl(H)**. Thus we get,
Edg Cl(H $\cup$ K) and **Edg Cl(H)** $\cup$ **Edg Cl(K)** are identical.

d) We have Suppose H and K be any two edge induced subgraphs of G. Now we have E(H$\cap$ K) = E(H)$\cap$ E(K). It gives,
(H $\cap$ K) $\subseteq$ H and (H $\cap$ K) $\subseteq$ K. Then by the above part (b),

$$\begin{gathered} \textbf{Edg Cl (H} \cap \textbf{K)} \subseteq \textbf{Edg Cl H}, \\ \textbf{Edg Int (H} \cap \textbf{K)} \subseteq \textbf{Edg Cl K} \\ \Longrightarrow \text{E}(\textbf{Edg Cl (H} \cap \textbf{K)}) \subseteq \text{E}(\textbf{Edg Cl H}). \\ \text{E}(\textbf{Edg Cl (H} \cap \textbf{K)}) \subseteq \text{E}(\textbf{Edg Cl K}). \end{gathered}$$

So we get,
E(**Edg Cl (H $\cap$ K)**) $\subseteq$ E(**Edg Cl H**) $\cap$ E(**Edg Cl K**).
Edg Cl (H $\cap$ K) $\subseteq$ **Edg Cl H** $\cap$ **Edg Cl K** .

□

6. Conclusion

In this work, we determined some topological propreties on graph structures, in particular edge induced subgraphs. In future anyone can verify the other known topological parameters with this graph topological structure.

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