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FORMULATION OF DIFFERENTIAL EQUATIONS UTILIZING THE RELATIONSHIP AMONG RAMANUJAN-TYPE EISENSTEIN SERIES AND h-FUNCTIONS

VIDYA H C, PRAGATHI B SHETTY, AND ASHWATH RAO B*

Abstract. Ramanujan, in his lost notebook on page 188, delineated unique categories of remarkable infinite series, expressing them through the framework of Eisenstein series. The aim of this paper is to explore diverse differential identities concerning classical η-functions and h-functions. Furthermore, we introduce specific identities involving Eisenstein series of various levels and h-functions, which serve to establish connections between class one infinite series and h-functions.

1. Introduction

Differential equations play a pivotal role in applied mathematics, serving as fundamental tools in the advancement of clinical, engineering, physics, and chemistry disciplines. In homage to Ramanujan's contributions, we endeavor to construct ordinary differential equations by utilizing derived Eisenstein series relations of different levels. Significant research endeavors have been directed towards ordinary differential equations that are fulfilled by modular forms. In his notebook referenced as [1], Ramanujan dedicated considerable focus to Eisenstein series, particularly to P, Q, and R, and provided several intriguing differential identities involving infinite series and theta functions.

The primary focus of this paper lies in the development of numerous differential identities that incorporate theta functions and an infinite product, specifically referred to as h-functions. A detailed study of h-functions and numerous modular equations for h has been derived by M. S. M. Naika et.al [8]. In their research, S. Cooper and D. Ye [4] undertake a thorough examination of the h-function and illustrate its applicability through the presentation of numerous elegant formulas involving this parameter. Furthermore, S. Cooper [5] has documented specific relationships between Eisenstein series and h-functions. Ramanujan [1] documented certain differential equations related to η-functions, while B. C. Berndt [3] extensively emphasized the significance of constructing differential equations that incorporate η-functions and Eisenstein series. In their referenced work [7], H. C. Vidya and B. Ashwath Rao established graceful linkages between Eisenstein series and theta functions. These connections were later utilized to deduce and assess particular convolution identities. Additionally, H. C. Vidya and B. R. Srivatsa Kumar, as cited in [10], derived differential equations integrating identities associated

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with the η-function. They emphasized the importance of devising these equations to facilitate the generation of incomplete integrals containing η-functions.
Furthermore, on page 188 of his missing notebook referenced as [9], Ramanujan derived specific formulas connecting the class one infinite series $T_{2r}(q)$, where $r = 1, 2, ..., 6$, with the Eisenstein series P, Q, and R. The primary proof of six formulas for $T_{2r}(q)$ appeared in a research document by B. C. Berndt and A. J. Yee [2]. Another proof of these formulas appeared in Liu's paper [6, p. 9-12].
Section 2 is dedicated to documenting preliminary discoveries aimed at facilitating the attainment of the primary objectives. In Section 3, a correlation has been established between Eisenstein series and h-functions, which is subsequently employed to derive specific differential identities and establish relationships among Class one infinite series and h-functions.

2. Preliminaries

Ramanujan presented the following definition of a general theta function in his notebook [1, p.35]. Consider any complex a, b and q, with $|ab| < 1$,

$$f(a,b) := \sum_{i=-\infty}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty,$$

where,

$$(a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i), \quad |q| < 1.$$

The subsequent instances are specific cases of theta functions as defined by Ramanujan [1, p.35].:

$$\varphi(q) := f(q,q) = \sum_{i=-\infty}^{\infty} q^{i^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

$$f(-q) := f(-q,-q^2) = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2}$$

$$= (q;q)_\infty = q^{-1/24}\eta(\tau),$$

where $q = e^{2\pi i\tau}$.
We denote $f(-q^n) = f_n$.

Definition 2.1. Cooper [5] documented certain relations involving η-functions in his notebook, as outlined below:

$$\begin{aligned} r_a(q) &= \frac{\eta_2\eta_6^5}{\eta_1^5\eta_3},\ r_b(q) = \frac{\eta_1^2\eta_6^4}{\eta_2^4\eta_3^2},\ r_c(q) = \frac{\eta_1\eta_6^3}{\eta_2\eta_3^3},\\ 1+8r_a(q) &= \frac{\eta_2^4\eta_3^8}{\eta_1^8\eta_6^4},\ 1+9r_a(q) = \frac{\eta_2^9\eta_3^3}{\eta_1^9\eta_6^3},\\ 1-3r_b(q^2) &= \frac{\eta_1^3\eta_{12}}{\eta_3\eta_4^3},\ 1+r_b(q^2) = \frac{\eta_2^2\eta_3^e ta_{12}}{\eta_1\eta_4^3\eta_6^2}. \end{aligned}$$

Definition 2.2. Ramanujan[9] documented the class one infinite series,

$$T_{2k}(q) := 1+\sum_{x=1}^{\infty}(-1)^x\Big[(6x-1)^{2k}\{q^{\frac{x(3x-1)}{2}} +(6x+1)^{2k}q^{\frac{x(3x+1)}{2}}\}\Big], \tag{2.1}$$

and expressed $T_{2k}(q)$ for $k=1,2,...,6$ in terms of Ramnujan-type Eisenstein series:

$$\begin{aligned} P(q) &:= 1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j}\\ Q(q) &:= 1+240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j} = 1+240\sum_{j=1}^{\infty}\delta_3(j)q^j,\\ R(q) &:= 1-504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j} = 1-504\sum_{j=1}^{\infty}\delta_5(j)q^j. \end{aligned}$$

Throughout, we denote $P(q^n) = P_n$.

Further, B. C. Berndt [2] established an interesting relation

$$T_2(q) = (q;q)_\infty P(q). \tag{2.2}$$

Definition 2.3. [5] For $|q|<1$, the h-function defined by

$$h := h(q) := q\prod_{x=1}^{\infty}\frac{(1-q^{12x-1})(1-q^{12x-11})}{(1-q^{12x-5})(1-q^{12x-7})}.$$

The weight two modular form y_{12} in terms of h-function is defined by

$$y_{12} = q\frac{d}{dq}logh = 1-\sum_{s=1}^{\infty}\chi_{12}(s)\frac{sq^s}{1-q^s},$$

where

$$\chi_{12}(s) = \begin{cases} 1 & \text{if s=1 or 11 (mod 12)},\\ -1 & \text{if s=5 or 7 (mod 12)},\\ 0 & \text{otherwise}. \end{cases}$$

Lemma 2.4. [5] *The relation among an infinite series and h-functions hold:*

$$\begin{pmatrix} P(q) \\ P(q^2) \\ P(q^3) \\ p(q^4) \\ P(q^6) \\ P(q^{12}) \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 & -6 & 2 & 0 \\ 3 & 2 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 2 & -2 & \frac{5}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{3}{2} & \frac{5}{4} & -\frac{5}{3} & \frac{3}{4} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{h(dy_{12})}{dh} \\ \frac{(1-h^2)}{(1+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-4h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1-2h+h^2)}y_{12} \\ \frac{(1-h^2)}{(1+2h+h^2)}y_{12} \end{pmatrix}.$$

3. Relations among Eisenstein series and h-functions

Theorem 3.1. *The connections among the Ramanujan type Eisenstein series and h-function holds:*

$$(i) - P_1 + P_2 + 3P_3 - 3P_6 = \frac{24v}{u(v-4)}y_{12}$$

$$(ii) - 5P_1 + 2P_2 - 3P_3 + 30P_6 = \frac{24uv}{(u-4)(u-2)}y_{12}$$

$$(iii)P_1 - 4P_2 - 3P_3 + 12P_6 = \frac{6u(u-1)v}{(u-1)(u-2)(u+2)}y_{12}$$

$$(iv)P_1 - 2P_2 - 9P_3 + 18P_6 = \frac{8(u-4)v}{u(u-2)}y_{12}$$

$$(v) - P_1 + 2P_2 + P_3 - 2P_6 = \frac{8uv}{(u-1)(u+2)(u-4)}y_{12}$$

$$(vi)P_1 - P_2 - 4P_4 + 4P_{12} = -\frac{24v}{(u-1)(u-4)}y_{12}$$

$$(vii) - P_1 + 4P_2 + 9P_3 - 12P_4 - 12P_6 + 12P_{12} = \frac{24v}{u(u-1)}y_{12}$$

$$(viii) - P_1 + 2P_2 + 3P_3 + -6P_6 = \frac{2v(-u^4 + 16u^3 - 24u^2 - 32u + 32)}{u(u-1)(u-2)(u-4)(u+2)}y_{12}$$

$$(ix)P_1 + 3P_3 - 4P_4 - 12P_6 = \frac{3(5u^2 + 13u - 20)v}{u(u-1)(u-4)}y_{12}$$

$$(x)P_1 - 2P_2 + 3P_3 - 6P_6 = \frac{3(-u^4 - 2u^3 + 12u^2 - 32u + 32)v}{u(u-1)(u-2)(u-4)(u+4)}y_{12}$$

$$(xi) -17P_1+26P_2+33P_3-42P_6=\frac{24v(17u^2+8u-16)}{u(u-4)(u+2)}y_{12}$$

$$(xii) P_1-10P_2+15P_3-6P_6=\frac{24v(2-u)}{u(u-1)(u+2)}y_{12}$$

$$(xiii) -11P_1+8P_2+27P_3+12P_4-24P_6-12P_{12}$$
$$=\frac{24(-u^3+16u^2-34u+16)v}{u(u-1)(u-4)(u-2)}y_{12}$$

$$(xiv) -9P_1+12P_2+33P_3-12P_4-36P_6+12P_{12}=\frac{36v(3u-4)}{u(u-1)(u-4)}y_{12}$$

$$(xv) -7P_1+4P_2+15P_3+12P_4-12P_6-12P_{12}=\frac{24v(7u-4)}{u(u-1)(u-4)}y_{12},$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$.

Proof. Through the utilization of Lemma 2.4 and subsequent simplification, we derive the requisite relationships. □

4. Formation of differential identities involving η-functions and h-functions

In succeeding Theorems 4.1−4.3, we formulate differential equations in consequence of Eisenstein series of different levels that associates h-functions and the derivative of weight two modular forms.

Theorem 4.1. *1. If*

$$S(q)=1+8r_a(q),$$

then the following differential identity holds:

$$q\frac{ds}{dq}-\left[\frac{8v}{u(v-4)}y_{12}\right]S=0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$.

Proof. We note from [5, pp. 189] that, $S(q)$ may be reformulated in terms of η-functions given by

$$S(q)=\frac{\eta_2^4\eta_3^8}{\eta_1^8\eta_6^4}=\frac{f_2^4f_3^8}{f_1^8f_6^4}.$$

Rewriting the expression for $S(q)$ using q-series notation, and then logarithmically differentiating with respect to q, we reach the following outcome:

$$\frac{1}{S}\frac{dS}{dq}=\frac{8}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}}-3\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}}+3\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\right].$$

By utilizing the definition of the Eisenstein series and performing simplifications, we obtain the following result:

$$\frac{q}{S}\frac{dS}{dq}=\frac{1}{3}[-P_1+P_2+3P_3-3P_6].$$

Finally, incorporating Lemma 2.4, and representing Eisenstein series in terms of h-functions, followed by simplification, we derive

$$\frac{q}{S}\frac{dS}{dq} = \frac{8h(1-h^2)}{(1+h^2)(1-4h+h^2)}.$$

Further, denoting $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and then simplifying, we derive expression (i).

Applying the same methodology used to derive the aforementioned result, we can construct the following set of differential equations:

(ii) If $S(q) = r_a(q) = \dfrac{qf_2f_6^5}{f_1^5f_3}$, then $q\dfrac{ds}{dq} - \left[\dfrac{uv}{(u-4)(u-2)}y_{12}\right]S = 0,$

(iii) If $S(q) = r_b(q) = \dfrac{q^{1/2}f_1^2f_6^4}{f_2^4f_3^2}$, then $q\dfrac{ds}{dq} - \left[\dfrac{1}{2}\dfrac{vu(u-1)}{(u-1)(u-2)(u+2)}y_{12}\right]S = 0,$

(iv) If $S(q) = r_c(q) = \dfrac{q^{1/3}f_1f_6^3}{f_2f_3^3}$, then $q\dfrac{ds}{dq} - \left[\dfrac{1}{3}\dfrac{v(u-4)}{u(u-2)}y_{12}\right]S = 0,$

(v) If $S(q) = 1+9r_a(q) = \dfrac{f_2^9f_3^3}{f_1^9f_6^3}$, then $q\dfrac{ds}{dq} - \left[\dfrac{8uv}{(u-1)(u+2)(u-4)}y_{12}\right]S = 0,$

(vi) If $S(q) = 1-3r_b(q^2) = \dfrac{f_1^3f_{12}}{f_3f_4^3}$, then $q\dfrac{ds}{dq} + \left[\dfrac{3v}{(u-1)(u-4)}y_{12}\right]S = 0,$

(vii) If $S(q) = 1+r_b(q^2) = \dfrac{f_2^2f_3^3f_{12}}{f_1f_4^3f_6^2}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v}{u(u-1)}y_{12}\right]S = 0,$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$. Therefore the proof is completed. □

Theorem 4.2. *If*

$$S(q) = \frac{1}{r_a(q)} + 72r_a(q) + 17,$$

then the following differential identity holds:

$$q\frac{ds}{dq} - \left[\frac{v(-u^4+16u^3-24u^2-32u+32)}{(u-1)(u-2)(u+2)(u-4)}y_{12}\right]S = 0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$.

Proof. According to the information provided in [5, p. 189], it appears that $S(q)$ can be expressed using theta functions as

$$S(q) = \frac{f_2^{12}f_3^{12}}{qf_1^{12}f_6^{12}},$$

By reformulating the expression for $S(q)$ in q-series notation and subsequently logarithmically differentiating with respect to q, we arrive at the following result:

$$\frac{1}{S}\frac{dS}{dq} = -\frac{1}{q} + \frac{12}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - \sum_{r=1}^{\infty}2\frac{rq^{2r}}{1-q^{2r}} - 3\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}} + 6\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\right].$$

Upon applying the definition of the Eisenstein series and conducting simplifications, we arrive at the following outcome:

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{2}[-P_1 + 2P_2 + 3P_3 - 6P_6].$$

Finally, incorporating Lemma 2.4 and representing Eisenstein series in terms of h-functions, followed by simplification, we derive expression (i).

Employing the methodology utilized in deriving the aforementioned results, we can formulate the following set of differential equations:

(ii) If $S(q) = \dfrac{1}{r_b(q^2)} - 3r_b(q^2) - 2 = \dfrac{f_1^2 f_3^2}{q f_4^2 f_{12}^2}$, then

$$q\frac{ds}{dq} - \left[\frac{1}{4}\frac{v(5u^2 + 13u - 20)}{u(u-1)(u-4)} y_{12}\right] S = 0,$$

(iii) If $S(q) = \dfrac{1}{x_b(q)} + 9x_b(q) - 10 = \dfrac{f_1^6 f_3^6}{q f_2^6 f_6^6}$, then

$$q\frac{ds}{dq} - \left[\frac{v(-u^4 - 2u^3 + 12u^2 - 32u + 32)}{u(u-1)(u-4)(u-2)(u+2)} y_{12}\right] S = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$. Therfore the proof is completed.

□

Theorem 4.3. *If*

$$S(q) = 1 + 17r_a(q) + 72r_a^2(q),$$

then the following differential identity holds:

$$q\frac{ds}{dq} - \left[\frac{v(17u^2 + 8u - 16)}{u(u-1)(u-4)(u+2)} y_{12}\right] S = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$.

Proof. Based on the details outlined in [5, p. 189], it appears that the expression of $S(q)$ can be represented utilizing theta functions.

$$S(q) = \frac{f_2^{13} f_3^{11}}{f_1^{17} f_6^7}$$

Through rephrasing the expression for $S(q)$ in q-series notation and subsequently applying logarithmic differentiation with respect to q, we obtain the following outcome:

$$\frac{1}{S}\frac{dS}{dq} = \frac{1}{q}\left[17\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - 26\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}} - 33\sum_{r=1}^{\infty}\frac{rq^{3r}}{1-q^{3r}} + 42\sum_{r=1}^{\infty}\frac{rq^{6r}}{1-q^{6r}}\right].$$

After employing the definition of the Eisenstein series and performing simplifications, we reach the following result:

$$\frac{q}{S}\frac{dS}{dq} = \frac{1}{2}[-17P_1 + 26P_2 + 33P_3 - 42P_6].$$

Ultimately, by incorporating Lemma 2.4 and expressing Eisenstein series in the context of h-functions, subsequent simplification yields the expression (i).

Employing the methodology utilized in deriving the aforementioned results, we can formulate the following set of differential equations:

(ii) If $S(q) = \dfrac{1+8r_a(q)}{1+9r_a(q)} = \dfrac{f_1 f_3^5}{f_2^5 f_6}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(2-u)}{u(u-1)(u+2)} y_{12}\right] S = 0,$

(iii) If $S(q) = \dfrac{1+8r_a(q)}{1-3r_b(q^2)} = \dfrac{f_2^4 f_3^9 f_4^3}{f_1^{11} f_6^4 f_{12}}$, then

$$q\frac{ds}{dq} - \left[\frac{v(-u^3+16^2-34u+16)}{u(u-1)(u-4)(u-2)} y_{12}\right] S = 0,$$

(iv) If $S(q) = (1+8r_a(q))(1+r_b(q^2)) = \dfrac{f_2^6 f_3^{11} f_{12}^9}{f_1^9 f_4^3 f_6^6}$ then

$$q\frac{ds}{dq} - \left[\frac{3v(3u-4)}{u(u-1)(u-4)} y_{12}\right] S = 0.$$

(v) If $S(q) = \dfrac{1+8r_a(q)}{1+r_b(q^2)} = \dfrac{f_2^2 f_3^5 f_4^3}{f_1^7 f_6^2 f_{12}^1}$, then $q\dfrac{ds}{dq} - \left[\dfrac{v(7u-4)}{u(u-1)(u-4)} y_{12}\right] S = 0.$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$. Hence the proof. □

5. Relations among Class one infinite series and h-functions

Drawing inspiration from the research conducted by B. C. Berndt [2], H. C. Vidya and B. Ashwath Rao [7] formulated a representation for Eisenstein series in terms of the classical Class one infinite series. Using this expression, we derive fascinating equations that establish connections between Class one infinite series and h-functions.

Lemma 5.1. [7] *For every $n \geq 2$, the subsequent relationship among the two distinct series holds:*

$$P(q^n) = 1 + nq^{n-1}\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty} - 1\right]. \tag{5.1}$$

Theorem 5.2. *The following series expansion among class one infinite series and h-functions hold:*

$$i)\frac{T_2(q)}{f_1} - 2q\frac{T_2(q^2)}{f_2} - 9q^2\frac{T_2(q^3)}{f_3} + 18q^5\frac{T_2(q^6)}{f_6} + 2q\left(1-\frac{1}{f_2}\right)$$
$$+ 9q^2\left(1-\frac{1}{f_3}\right) - 18q^5\left(1-\frac{1}{f_6}\right) + \frac{24v}{u(v-4)}y_{12} - 1 = 0,$$

$$ii)\frac{5T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}+180q^5\frac{T_2(q^6)}{f_6}+4q\left(1-\frac{1}{f_2}\right)$$
$$+9q^2\left(1-\frac{1}{f_3}\right)-180q^5\left(1-\frac{1}{f_6}\right)+\frac{24uv}{(u-4)(u-2)}y_{12}-167=0,$$

$$iii)\frac{T_2(q)}{f_1}-8q\frac{T_2(q^2)}{f_2}-9q^2\frac{T_2(q^3)}{f_3}+72q^5\frac{T_2(q^6)}{f_6}+8q\left(1-\frac{1}{f_2}\right)$$
$$+9q^2\left(1-\frac{1}{f_3}\right)-72q^5\left(1-\frac{1}{f_6}\right)-\frac{6u(u-1)v}{(u-1)(u-2)(u+2)}y_{12}+55=0,$$

$$iv)\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}-27q^2\frac{T_2(q^3)}{f_3}+108q^5\frac{T_2(q^6)}{f_6}+8q\left(1-\frac{1}{f_2}\right)$$
$$+27q^2\left(1-\frac{1}{f_3}\right)-108q^5\left(1-\frac{1}{f_6}\right)-\frac{8(u-4)v}{u(u-2)}y_{12}+77=0,$$

$$v)-\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+3q^2\frac{T_2(q^3)}{f_3}-12q^5\frac{T_2(q^6)}{f_6}-q\left(1-\frac{1}{f_2}\right)$$
$$-3q^2\left(1-\frac{1}{f_3}\right)+12q^5\left(1-\frac{1}{f_6}\right)-\frac{8uv}{(u-1)(u+2)(u-4)}y_{12}+1=0,$$

$$vi)\frac{T_2(q)}{f_1}-2q\frac{T_2(q^2)}{f_2}-16q^3\frac{T_2(q^3)}{f_4}+48q^{11}\frac{T_2(q^6)}{f_{12}}+2q\left(1-\frac{1}{f_2}\right)$$
$$+16q^3\left(1-\frac{1}{f_4}\right)-48q^{11}\left(1-\frac{1}{f_{12}}\right)-\frac{24v}{(u-1)(u-4)}y_{12}+77=0,$$

$$vii)-\frac{T_2(q)}{f_1}+8q\frac{T_2(q^2)}{f_2}+27q^2\frac{T_2(q^3)}{f_3}-48q^3\frac{T_2(q^6)}{f_4}-72q^5\frac{T_2(q^2)}{f_6}$$
$$+144q^{11}\frac{T_2(q^3)}{f_{12}}-8q\left(1-\frac{1}{f_2}\right)-27q^2\left(1-\frac{1}{f_3}\right)+48q^3\left(1-\frac{1}{f_4}\right)$$
$$+72q^5\left(1-\frac{1}{f_6}\right)-144q^{11}\left(1-\frac{1}{f_{12}}\right)-\frac{24v}{u(u-1)}y_{12}+59=0,$$

$$viii)-\frac{T_2(q)}{f_1}+4q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}-4q\left(1-\frac{1}{f_2}\right)$$
$$-9q^2\left(1-\frac{1}{f_3}\right)+35q^5\left(1-\frac{1}{f_6}\right)$$
$$-\frac{2v(-u^4+16u^3-24u^2-32u+32)}{u(u-1)(u-2)(u-4)(u+2)}y_{12}-23=0,$$

$$ix)\frac{T_2(q)}{f_1}+q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}-16q^3\frac{T_2(q^6)}{f_4}-72q^5\frac{T_2(q^3)}{f_6}+2q\left(1-\frac{1}{f_2}\right)$$
$$-9q^2\left(1-\frac{1}{f_3}\right)+16q^3\left(1-\frac{1}{f_4}\right)+72q^5\left(1-\frac{1}{f_6}\right)$$
$$-\frac{3(5u^2+13u-20)v}{u(u-1)(u-4)}y_{12}+77=0,$$

$$x)\frac{T_2(q)}{f_1}-4q\frac{T_2(q^2)}{f_2}+9q^2\frac{T_2(q^3)}{f_3}--36q^4\frac{T_2(q^6)}{f_4}+4q\left(1-\frac{1}{f_2}\right)$$
$$-9q^2\left(1-\frac{1}{f_3}\right)+36q^5\left(1-\frac{1}{f_6}\right)$$
$$-\frac{3(-u^4-2u^3+12u^2-32u+32)v}{u(u-1)(u-2)(u-4)(u+4)}y_{12}-31=0,$$

$$xi)-17\frac{T_2(q)}{f_1}+52q\frac{T_2(q^2)}{f_2}+99q^2\frac{T_2(q^3)}{f_3}-225q^5\frac{T_2(q^6)}{f_6}-52q\left(1-\frac{1}{f_2}\right)$$
$$-99q^2\left(1-\frac{1}{f_3}\right)+225q^5\left(1-\frac{1}{f_6}\right)-\frac{24v(17u^2+8u-16)}{u(u-4)(u+2)}y_{12}-74=0,$$

$$xii)\frac{T_2(q)}{f_1}-20q\frac{T_2(q^2)}{f_2}+45q^2\frac{T_2(q^3)}{f_3}-36q^5\frac{T_2(q^6)}{f_6}+20q\left(1-\frac{1}{f_2}\right)$$
$$-45q^2\left(1-\frac{1}{f_3}\right)+36q^5\left(1-\frac{1}{f_6}\right)-\frac{24v(2-u)}{u(u-1)(u+2)}y_{12}-11=0,$$

$$xiii)-11\frac{T_2(q)}{f_1}+16q\frac{T_2(q^2)}{f_2}+81q^2\frac{T_2(q^3)}{f_3}-48q^3\frac{T_2(q^6)}{f_4}-144q^5\frac{T_2(q^2)}{f_6}$$
$$-144q^{11}\frac{T_2(q^3)}{f_{12}}-16q\left(1-\frac{1}{f_2}\right)-81q^2\left(1-\frac{1}{f_3}\right)-48q^3\left(1-\frac{1}{f_4}\right)+$$
$$144q^5\left(1-\frac{1}{f_6}\right)+144q^{11}\left(1-\frac{1}{f_{12}}\right)$$
$$-\frac{24(-u^3+16u^2-34u+16)v}{u(u-1)(u-4)(u-2)}y_{12}-143=0,$$

$$xiv)-9\frac{T_2(q)}{f_1}+24q\frac{T_2(q^2)}{f_2}+99q^2\frac{T_2(q^3)}{f_3}-48q^3\frac{T_2(q^6)}{f_4}-216q^5\frac{T_2(q^2)}{f_6}$$
$$+144q^{11}\frac{T_2(q^3)}{f_{12}}-24q\left(1-\frac{1}{f_2}\right)-99q^2\left(1-\frac{1}{f_3}\right)+48q^3\left(1-\frac{1}{f_4}\right)$$
$$+216q^5\left(1-\frac{1}{f_6}\right)-144q^{11}\left(1-\frac{1}{f_{12}}\right)-\frac{36v(3u-4)}{u(u-1)(u-4)}y_{12}+3=0,$$

$$xv)-7\frac{T_2(q)}{f_1}+8q\frac{T_2(q^2)}{f_2}+45q^2\frac{T_2(q^3)}{f_3}+48q^3\frac{T_2(q^6)}{f_4}-72q^5\frac{T_2(q^2)}{f_6}$$
$$-144q^{11}\frac{T_2(q^3)}{f_{12}}-8q\left(1-\frac{1}{f_2}\right)-45q^2\left(1-\frac{1}{f_3}\right)-48q^3\left(1-\frac{1}{f_4}\right)$$
$$+72q^5\left(1-\frac{1}{f_6}\right)+144q^{11}\left(1-\frac{1}{f_{12}}\right)-\frac{24v(7u-4)}{u(u-1)(u-4)}y_{12}-115=0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$ *and* $h\frac{dy_{12}}{dh} = w$.

Proof. By utilizing equations (2.2) and (5.1) within the context of Theorem 3.1 (i) to (xv), and subsequently streamlining the expressions, we arrive at equivalences involving $P(q^n)$ expressed in relation to $T_2(q^n)$ and h-functions, as detailed above. □

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Vidya H C:Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India.
Email address: vidya.rao@manipal.edu

Pragathi B. Shetty:Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India.
Email address: pragthishetty@gmail.com

Ashwath Rao B: Computer Science and Engineering Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576104, Karnataka, India.
Email address: ashwath.rao.b@gmail.com