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DETERMINANT OF ANTICHAIN GRAPHS

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ABSTRACT. Networks, in the form of bipartite graphs, are abundant in existence. A network having no redundant connections has proved to be more efficient and cost-effective too. Antichain graphs are a class of bipartite graphs having no redundant connections. Matrices provide models for graphs that illuminate their structure. The determinant is one of the powerful linear algebraic tools, that has been used extensively to study graphs. In this article, two of the powerful linear algebraic parameters namely, the determinant and permanent of adjacency matrix of antichain graphs are studied. This article characterizes the antichain graphs with nonzero determinants, which in turn, are the graphs with positive nullity.

Throughout the article, a bipartite graph G with the bipartition $V(G) = V_1 \cup V_2$ and $E(G) = E$ is denoted by $G(V_1 \cup V_2, E)$. For any two vertices u and v, we write $u \sim v$ if u is adjacent to v, $u \nsim v$ if they are not. The neighborhood of a vertex $u \in V(G)$ is denoted by $N_G(u)$. A dominating vertex in a graph G is a vertex which is adjacent with other vertices of of the graph. But, in the context of a bipartite graph $G(V_1 \cup V_2, E)$, a dominating vertex $u \in V_i$ is a vertex which is adjacent with all other vertices of V_j, for $i = 1, 2$ and $i \neq j$. The adjacency matrix $A = [a_{ij}]$ is the $n \times n$ matrix in which $a_{ij} = 1$ if $v_i \sim v_j$ and $a_{ij} = 0$ otherwise. We write $det(G)$ for the determinant of adjacency matrix of G. For a bipartite graph $G(V_1 \cup V_2, E)$, the adjacency matrix can be written as

$$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix},$$

where B is called the biadjacency matrix [1].

Antichain graphs. A class of sets $S = \{S_1, S_2, ..., S_n\}$ is called an antichain with respect to the operation of set inclusion if for every $S_i, S_j \in S$, neither $S_i \subseteq S_j$ nor $S_j \subseteq S_i$. A bipartite graph $G(V_1 \cup V_2, E)$ is called an antichain graph if neighborhoods of vertices of G form an antichain with respect to the operation of set inclusion.

Example 0.1. Consider the graph $G(V_1 \cup V_2, E)$ as shown in Figure 1, where $V_1 = \{v_1, v_3, v_5, v_7\}$ and $V_2 = \{v_2, v_4, v_6, v_8, v_9\}$ and the neighborhood of vertices given by $N_G(v_1) = \{v_2, v_8\}, N_G(v_2) = \{v_1, v_3\}, N_G(v_3) = \{v_2, v_4, v_9\} N_G(v_4) = \{v_3, v_5\}, N_G(v_5) = \{v_4, v_6\}, N_G(v_6) = \{v_5, v_7\}, N_G(v_7) = \{v_6, v_8, v_9\}, N_G(v_8) =$

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$\{v_1, v_7\}, N_G(v_9) = \{v_3, v_7\}$.

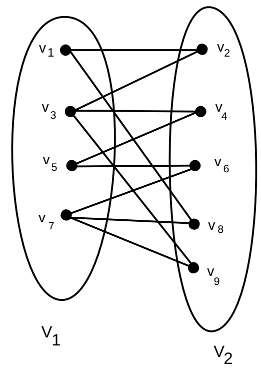

FIGURE 1. An antichain graph on 9 vertices

It can be noted that neither $N_G(v_i) \subseteq N_G(v_j)$ nor $N_G(v_j) \subseteq N_G(v_i)$ for every $1 \leq i, j \leq 9$ and $i \neq j$.

Authors of [2] clearly investigated the structure antichain graphs, using which generated large network of antichain graphs. Some of the properties are listed below.

(1) An antichain graph does not have isolated vertices.
(2) Let $G(V_1 \cup V_2, E)$ be a connected antichain graph where $G \neq K_2$. Then G contains neither a pendant vertex nor a dominating vertex.
(3) The graph K_2 is the only tree which is an antichain graph.
(4) Out of all the graphs with less than or equal to seven vertices, the complete graph K_2, the cycle graph C_6 and their possible vertex disjoint unions are the only antichain graphs.
(5) For a bipartite graph $G(V_1 \cup V_2, E)$, the $2-$complement G_2^P is an antichain graph with respect to the same bipartition $P = \{V_1, V_2\}$ if and only if G itself is an antichain graph.
(6) Let $G(V_1 \cup V_2, E)$ be a connected antichain graph on $2n(n \neq 2)$ vertices having m edges. Then $2n \leq m \leq n(n-1)$.
(7) Let $G(V_1 \cup V_2, E)$ be a connected antichain graph on $2n+1(n > 3)$ vertices having m edges. Then $m \geq 2n + 2$.

Matrix representation of graphs. Matrix representation of graphs allowing the use of simple yet powerful linear algebraic techniques to investigate them. The determinant, permanent, rank and eigenvalues are few of the powerful linear algebraic tools, which have been used extensively to study graphs. In specific, the parameters associated with the adjacency matrix of graphs are studied more extensively. For a graph G, the notations $rank(G)$, $det(G)$, $spec(G)$ and $per(G)$ describe the rank, determinant, eigenvalues and permanent of adjacency matrix of G respectively. If $\mu_1, \mu_2, \ldots, \mu_k$ are eigenvalues of the adjacency matrix of a graph G with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $spec(G)$ can be written as

$spec(G) = \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_k \\ m_1 & m_2 & \dots & m_k \end{pmatrix}.$

A subgraph G_1 of a graph G is said to be elementary if every component of G_1 is a cycle or an edge. The following theorem gives the expressions for determinant and permanent of a graph in terms of its elementary spanning subgraphs ([3], [4]).

Theorem 0.2. [4] *Let G be a graph on n vertices. Then,*

$$\det(G) = \sum_{G_1} (-1)^{n-k_1(G_1)-k_2(G_1)} 2^{k_2(G_1)}, \tag{0.1}$$

$$per(G) = \sum_{G_1} 2^{k_2(G_1)}, \tag{0.2}$$

where G_1 is the elementary spanning subgraph of G, $k_1(G_1)$ and $k_2(G_1)$ are the number of components in G_1 which are edges and cycles respectively.

Some more properties of determinants and permanents of graphs are discussed in [5]. Readers are referred to [6] for all the terminologies used, but not described in this article.

1. Determinant of Antichain graphs

An antichain graph on $2n$ vertices has at most $n(n-1)$ edges and the graph attaining the upper bound is the unique graph (up to isomorphism) obtained by removing a one factor from the complete bipartite graph $K_{n,n}$. Let this graph be denoted by $K^-_{n,n}$. An antichain graph $K^-_{4,4}$ is shown in Figure 2.

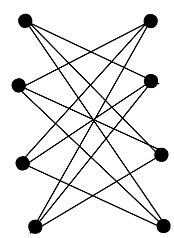

FIGURE 2. Antichain graph $K^-_{4,4}$

Theorem 1.1. *Let $K_{n,n}$ be an antichain graph on $2n$ vertices. Then*

$$det(K^-_{n,n}) = (-1)^n (n-1)^2 \tag{1.1}$$

$$per(K^-_{n,n}) = D_n^2 \tag{1.2}$$

$$spectrum(K^-_{n,n}) = \begin{cases} (n-1) & multiplicity\ 1 \\ (1-n) & multiplicity\ 1 \\ 1 & multiplicity\ (n-1) \\ -1 & multiplicity\ (n-1) \end{cases} \tag{1.3}$$

where $D_n = n!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + ... + \frac{(-1)^n}{n!}\right)$.

Proof. After relabeling the vertices of $K^-_{n,n}$, the adjacency matrix $A(K^-_{n,n})$ can be written as

$$A(K^-_{n,n}) = \left(\begin{array}{c|c} B_{n\times n} & 0_{n\times n} \\ \hline 0_{n\times n} & B^T_{n\times n} \end{array}\right)$$

where B is the biadjacency matrix of order n in which every entry is one except the diagonal entries and 0 is the block matrix in which every entry is zero. It is clear that $det(K^-_{n,n}) = (-1)^n det(B)^2$. Since $det(B) = (-1)^{n-1}(n-1)$, $det(K^-_{n,n}) = (-1)^n(n-1)^2$. Similarly, $per(K^-_{n,n}) = per(B)^2 = D_n^2$. Since $K^-_{n,n}$ is a regular bipartite graph with regularity $(n-1)$, it is true that $(n-1)$ and $(1-n)$ are eigenvalues. Further, one can easily note that $1, -1$ are eigenvalues each with multiplicities $(n-1)$. □

Theorem 1.2. *Let G be an antichain graph on $n \leq 7$ vertices. Let $\lambda \in spec(G)$. Then*

$$\lambda \in \{\pm 2, \pm 1\}$$

$$det(G) \in \{\pm 1, -4\} and$$

$$per(G) \in \{1, 4\}.$$

Proof. When $n = 3, 5, 7$, there is no antichain graph on n vertices. Further, the graph K_2 is the only antichain graph on 2 vertices having spectrum $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.
Similarly, when $n = 4$, the graph $K_2 \cup K_2$ is the only antichain graph on n vertices, whose spectrum is given by $\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$.
When $n = 6$, the graphs C_6 and $K_2 \cup K_2 \cup K_2$ are the only antichain graphs, spectrum of which are given by $\begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ 3 & 3 \end{pmatrix}$, respectively. The determinant follows from the fact that the product of eigenvalues gives the determinant. Further, on enumerating the elementary spanning subgraphs, we get the permanent. □

Theorem 1.3. *Let G be an antichain graph on 8 vertices. Let $\lambda \in spec(G)$. Then,*

$$\lambda \in \{\pm 3, \pm 2, \pm\sqrt{2}, \pm 1, 0\} \text{ and}$$

$$det(G) \in \{1, 4, 0, 9\}.$$

Proof. Let G be an antichain graph on 8 vertices. Then G is one of the following.

(1) $K_2 \cup K_2 \cup K_2 \cup K_2$
(2) $C_6 \cup K_2$
(3) C_8
(4) Hypercube Q_3 on 8 vertices and 12 edges.

When $G = K_2 \cup K_2 \cup K_2 \cup K_2$, $spec(G) = \begin{pmatrix} -1 & 1 \\ 4 & 4 \end{pmatrix}$. When $G = C_6 \cup K_2$, $spec(G) = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 3 & 3 & 1 \end{pmatrix}$. When $G = C_8$, $spec(G) = \begin{pmatrix} -2 & -\sqrt{2} & 0 & \sqrt{2} & 2 \\ 1 & 2 & 2 & 2 & 1 \end{pmatrix}$. Similarly, when $G = Q_3$, $spec(G) = \begin{pmatrix} -3 & -1 & 1 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$.

□

Theorem 1.4. *Let G be a unicyclic antichain graph on n vertices. Then*

$$per(G) = 4 \tag{1.4}$$

$$det(G) = \begin{cases} 0 & \frac{n}{2} \text{ is even} \\ -4 & \frac{n}{2} \text{ is odd} \end{cases}. \tag{1.5}$$

Proof. Since even cycles are the only unicyclic antichain graphs, n is even and $n > 4$. Thus $per(C_n) = 4$ and $det(C_n) = \begin{cases} 0 & \frac{n}{2} \text{ is even} \\ -4 & \frac{n}{2} \text{ is odd.} \end{cases}$. □

Theorem 1.5. *Let G be a connected antichain graph with exactly two cycles. Then $det(G) \in \{0, \pm 16\}$ and $per(G) = 16$.*

Proof. The connected antichain graph G with exactly two cycles is obtained by either of the following:

(1) Identifying any two vertices $u \in V(C_1), v \in V(C_2)$ of the cycles C_1 and C_2 of even length.
(2) Joining any two vertices $u \in V(C_{2k}), v \in V(C_{2l})$ of the two cycles C_{2k} and C_{2l} of even length by a path $P_r (r \geq 2, k, l \neq 2)$ of length r.

In the first type, since the cycles C_1 and C_2 are of even length, the number of vertices in the resultant graph G is odd. Thus G has no elementary spanning subgraphs and hence $det(G) = per(G) = 0$. In the second type, if G is obtained by joining any two vertices $u \in V(C_{2k}), v \in V(C_{2l})$ of the cycles C_{2k} and C_{2l} of even length by a path $P_r \equiv (u = w_1, w_2, ..., w_{k-1}, w_r = v)$, when r is odd, the graph G has odd number of vertices and hence $det(G) = per(G) = 0$. When r is even, we get the following elementary spanning subgraphs:

(1) $C_{2k} \cup C_{2l} \cup \frac{(r-2)}{2}$ K_2s
(2) $C_{2k} \cup \left(l + \frac{(r-2)}{2}\right)$ K_2s (2 such graphs)
(3) $C_{2l} \cup \left(k + \frac{(r-2)}{2}\right)$ K_2s (2 such graphs)
(4) $\left(k + l + \frac{(r-2)}{2}\right)$ K_2s (4 such graphs)

Using the expression for the determinant and the permanent of a graph given in 0.2, we evaluate the permanent.

$$per(G) = 2^2 + 2(2 + 2) + 4(2^0) = 16.$$

In the process of evaluation of determinant, we consider all possible cases and the expression for $det(G)$ in Theorem 0.2 yields as follows:

(1) When k, l are odd and $\frac{r}{2}$ is even: $det(G) = -16$
(2) When k, l are odd and $\frac{r}{2}$ is odd: $det(G) = 16$

(3) When k, l are even and $\frac{r}{2}$ is even: $det(G) = 0$
(4) When k, l are even and $\frac{r}{2}$ is odd: $det(G) = 0$
(5) When k is odd, l is even and $\dfrac{r}{2}$ is even: $det(G) = 0$
(6) When k is odd, l is even and $\dfrac{r}{2}$ is odd:$det(G) = 0$

□

The graphs discussed in the proof of Theorem 1.5 are as shown in Figure 3.

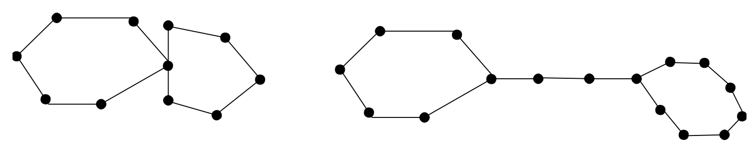

FIGURE 3. Antichain graphs obtained by identifying vertices of $C_6,\ C_6$ (left) and joining vertices of $C_6,\ C_8$ by P_4 (right)

Lemma 1.6. *Let G be a graph on n vertices. Then $det(G) = per(G)(\neq 0)$ if and only if, for all the elementary spanning subgraphs of G, n and $k_1 + k_0$ have the same parity where k_1, k_0 are the number of components in the elementary spanning subgraphs which are cycles and $K_2 s$ respectively.*

Proof. From Theorem 0.2, $det(G) = per(G)(\neq 0)$ if and only if, all the terms in the summation of $det(G)$ are positive. This is possible either when $n = k_1 + k_0$ or when both n and $k_1 + k_0$ have same parity. But, it is obvious that, for any elementary spanning subgraph H of G, $n \neq k_1 + k_0$. □

For any bipartite graph G on odd number of vertices, $per(G) = det(G) = 0$ as there are no elementary spanning subgraphs. Further, for a bipartite graph G, if $per(G) = det(G) \neq 0$, then the number of vertices in G must be even. Also, for any bipartite graph on even number of vertices, $per(G) = 0$ if and only if G has no elementary spanning subgraphs. From these facts and Lemma 1.6, one can note the following.

Lemma 1.7. *Let G be a antichain graph on n vertices. Then $per(G) = det(G) \neq 0$ if and only if $k_1 + k_0$ is even for all the elementary spanning subgraphs of G.*

The above statement is true even for bipartite graphs, rather than antichain graphs.

Theorem 1.8. *Let G be an antichain graph obtained by addition of a chord to a cycle $C_{2n}, n > 2$ such that it induces two more cycles C_k and C_l where ($k, l \neq 4$ and k, l are even, $k + l = 2n - 2$) has $det(G) \in \{\pm 1, 9\}$ and $per(G) = 9$.*

Proof. The graph G has the following elementary spanning subgraphs:
(1) C_{2n}
(2) $n\ K_2 s$ (3 such graphs)
(3) $C_k \cup \dfrac{(l-2)}{2}\ K_2 s$ (2 such graphs)

(4) $C_l \cup \dfrac{(k-2)}{2}\ K_2s$ (2 such graphs)

Form Theorem 0.2, $per(G) = 9$. For evaluation of determinant, we consider the two cases when n is even and n is odd separately.
When n is even, $n = \dfrac{k+l+2}{2} = \dfrac{k+1}{2} - 1$ is even, which implies $\dfrac{k+1}{2}$ is odd. This is possible when only one of $\dfrac{k}{2}$ or $\dfrac{l}{2}$ is even. Thus, when n is even, $\dfrac{k}{2}$ is odd and $\dfrac{l}{2}$ is even, thus $det(G) = 1$. Same is the case when $\dfrac{k}{2}$ is even and $\dfrac{l}{2}$ is odd.
When n is odd, $\dfrac{k+l}{2}$ is even. We consider the following cases.

(1) When $\dfrac{k}{2}, \dfrac{l}{2}$ both are even: $det(G) = -1$.
(2) When $\dfrac{k}{2}, \dfrac{l}{2}$ both are odd: $det(G) = -9$.

Hence $det(G) \in \{1, -1, -9\}$. □

Figure 4 represents an antichain graph discussed in Theorem 1.8.

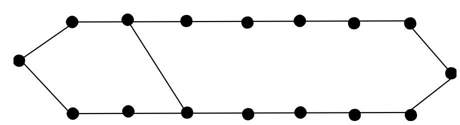

FIGURE 4. Antichain graphs obtained by addition of a chord in C_{16} such that it induces C_6 and C_{12}

Conclusion: The nullity of a graph denoted by is the multiplicity of 0 in the spectrum of G. Nullity of a molecular graph has good applications in quantum chemistry and Hückel molecular orbital theory. Characterizing graphs of positive nullity is an open problem which has attracted attention of researches in the field of chemical graph theory. In [7], the authors gave some of the significant results on graphs with nullity 0 or 1. Clearly, a graph has positive nullity if and only if its determinant is zero. This article discusses about the determinant of antichain graphs, which in turn contributes towards nullity.

Scope for future work: Antichain graphs are defined purely motivated by the structure of chain graphs. Since chain graphs are bipartite, naturally the restriction of bipartiteness is retained even in antichain graphs. But the generalized version, where the condition of bipartiteness is omitted, can be defined as a graph in which the neighborhoods of vertices form an antichain with respect to the operation of set inclusion. This class contains graphs like odd cycles which are otherwise not antichain graphs. This generalized class may be studied extensively.

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