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REMARKABLE ASSOCIATIONS AMONG THE INFINITE SERIES $N(q)$ AND η-FUNCTIONS

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ABSTRACT. This study aims to explore the relationships between Ramanujan-type Eisenstein series and the Borwein cubic theta functions, utilizing the (p, k)-parametrization technique proposed by Alaca, which provides a novel framework for deriving new Eisenstein series identities that include the Borwein theta functions. This paper presents fresh Eisenstein series identities involving Borwein's cubic theta functions, obtained using the (p, k)-parametrization technique.

1. Introduction

In this article, we explore the relationships between Eisenstein series and Borwein's cubic theta functions using parameters introduced by Alaca. Eisenstein series are complex valued functions central to number theory and the theory of modular forms, while Borwein's cubic theta functions are special functions associated with cubic forms and modular forms. The parameters p and k introduced by Alaca play a crucial role in revealing the connections between these two mathematical constructs. Through a detailed examination of these parameters, we uncover significant relationships between Eisenstein series and Borwein's cubic theta functions. What sets this study apart is that it extends beyond purely computational techniques. We provide an analytical demonstration that certain Ramanujan-type Eisenstein series special classes of Eisenstein series with exceptional properties can be expressed as combinations of Borwein's cubic theta functions. This not only illuminates the complex interaction between these mathematical objects but also offers deeper insights into the structures and symmetries inherent in modular forms and related mathematical fields.

Section 2 provides the essential background and insights needed to support the main objectives of the article, serving as a foundation for the concepts, theories, and methodologies discussed in the following sections. These preliminary insights lay a solid foundation for developing further arguments, theories, or analyses, including key definitions, background information, prior research, and theoretical frameworks. Section 3 presents a set of intriguing new identities, inspired by but

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distinct from Earnest Xia's earlier work. These identities uncover novel mathematical relationships discovered through our research, including the Ramanujan-Eisenstein series and Borwein's cubic theta functions. By revealing these identities, we enhance mathematical understanding and highlight connections between key concepts. This section serves as a platform for our contributions, potentially inspiring further research in related areas.

2. Preliminaries

The origins of the arithmetic-geometric mean iteration are deeply intertwined with the study of elliptic functions and theta functions. In groundbreaking work, the Borwein brothers [3, 4] uncovered a class of multidimensional theta functions, laying a crucial groundwork for further investigations.

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2}.$$
$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2}.$$
$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2}.$$

for $|q| < 1$, where q represents complex numbers, and $\omega = exp(2\pi i/3)$ is the principal cube root of unity, the given expressions for two-dimensional theta functions reveal that when $q = 0$, the values become $a(q) = 1$, $b(q) = 1$, and $c(q) = 0$.

Euler's binomial theorem, which expands expressions of the form $(1 + x)^n$ for any real number n, served as a foundational tool in the work of the Borwein siblings. They utilized this theorem to derive infinite product representations for the functions $b(q)$ and $c(q)$.

By applying Euler's binomial theorem, the Borwein siblings have likely formulated expressions for $b(q)$ and $c(q)$ as products of terms from binomial expansions, possibly with coefficients that follow specific patterns. These representations are expected to hold mathematical significance and provide insights into the behavior and properties of the functions $b(q)$ and $c(q)$.

Utilizing Euler's binomial theorem as a starting point, the Borwein siblings have formulated representations for both $b(q)$ and $c(q)$ in the form of infinite products, as demonstrated below:,

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},$$
$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i).$$

The function $a(q)$ can be expressed as follows (J.M. Borwein & P. B. Borwein [3]) and Berndt [2]:

$$a(q) = (-q;q^2)^2_\infty(q^2;q^2)_\infty(-q^3;q^6)^2_\infty(q^6;q^6)_\infty + 4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

Additionally, they have deduced the basic cubic identity that expresses the relationship between $a(q)$, $b(q)$, and $c(q)$ as follows:

$$a^3(q) = b^3(q) + c^3(q).$$

Definition 2.1. In his notebook [7], Srinivasa Ramanujan elucidated the definitions of the Eisenstein Series $L(q)$, $M(q)$ and $N(q)$ as outlined below:

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}.$$
$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r}.$$
$$N(q) := 1 - 504\sum_{r=1}^{\infty}\frac{r^5q^r}{1-q^r}.$$

Definition 2.2. For any complex c and d, Ramanujan[2, p.35] documented a general theta function,

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2}$$
$$:= (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q|<1.$$

The special case of theta function defined by Ramanujan[2, p.35],

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty.$$

$$f(-q) := f(-q,-q^2) = \sum_{i=-\infty}^{\infty}(-1)^i q^{i(3i-1)/2} = (q;q)_\infty = q^{-1/24}\eta(\tau),$$

where $q = e^{2\pi i\tau}$.

In their influential work, Alaca et al. [1] presented the $(p,\ k)$ parametrization of theta functions, which plays a crucial role in the development of the duplication and triplication principles. This approach facilitates the derivation of specific sum-to-product identities. The parameters p and k are defined in the following manner:

$$p := p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\phi^2(q^3)}.$$
$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

Lemma 2.3. [1] *Concerning the previously mentioned Eisenstein series* [3, 4]*, the expressions for $M(q)$, $N(q)$, $M(q^l)$, $N(q^l)$, $L(q) - lL(q^l)$, where $(l = 2, 3, 4, 6, 12)$, as well as $L(-q^l) - rL(q^r)$, where $l \in 1, 3$ and $r \in 1, 2, 3$, in terms of the parameters p and k, are articulated as follows:*

$$M(q) = (1 + 124p(1 + p^6) + 964p^2(1 + p^4) + 2788p^3(1 + p^2) + 3910p^4 + p^8)k^4,$$
$$M(q^2) = (1 + 4p(1 + p^6) + 64p^2(1 + p^4) + 178p^3(1 + p^2) + 235p^4 + p^8)k^4,$$
$$M(q^3) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) + 28p^3(1 + p^2) + 70p^4 + p^8)k^4,$$
$$M(q^6) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) - 2p^3(1 + p^2) - 5p^4 + p^8)k^4,$$
$$M(q^{12}) = (1 + 4p(1 + p) - 2p^3(1 + p^2) - 5p^4 + p^6(1 + p)/4 + p^8/16)k^4,$$
$$L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2,$$
$$L_{1,2}(q) = (L(-q) - L(q))/48 = (p/2 + 3p^2/4 + 3p^3/8 + p^4/16)k^2,$$
$$L_{1,2}(q^3) = (L(-q^3) - L(q^3))/48 = p^3(2 + p)k^2/16,$$
$$L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,$$
$$L(q) - 2L(q^2) = -(1 + 14p(1 + p^2) + 24p^2 + p^4)k^2,$$
$$L(q) - 3L(q^3) = -(1 + 8p(1 + p^2) + 18p^2 + p^4)k^2,$$
$$L(q) - 6L(q^6) = -(5 + 22p(1 + p^2) + 36p^2 + 5p^4)k^2,$$
$$L(q^2) - 3L(q^6) = -2(1 + 2p(1 + p^2) + 3p^2 + p^4)k^2,$$
$$L(q^3) - 2L(q^6) = -(1 + 2p(1 + p^2) + p^4)k^2,$$
$$L(q) - 4L(q^4) = -3(1 + 6p + 12p^2 + 8p^3)k^2,$$
$$L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2,$$
$$N(q) = (1 - 246p - 5532p^2 - 38614p^3 - 135369p^4 - 276084p^5 - 348024p^6$$
$$- 276084p^7 - 135369p^8 - 38614p^9 - 5532p^{10} - 246p^{11} + p^{12})k^6,$$
$$N(q^2) = (1 + 6p - 114p^2 - 625p^3 - \frac{4059}{2}p^4 - 4302p^5 - 5556p^6$$
$$- 4302p^7 - \frac{4059}{2}p^8 - 625p^9 - 114p^{10} + 6p^{11} + p^{12})k^6,$$
$$N(q^3) = (1 + 6p + 12p^2 - 58p^3 - 297p^4 - 396p^5 - 264p^6 - 396p^7 - 297p^8$$
$$- 58p^9 + 12p^{10} + 6p^{11} + p^{12})k^6.$$

$$N(q^6)=(1+6p+12p^2+5p^3-\frac{27}{2}p^4-18p^5-12p^6-18p^7-\frac{27}{2}p^8+5p^9$$
$$+12p^{10}+6p^{11}+p^{12})k^6.$$

Lemma 2.4. *Alaca et al.* [1] *have derived the parametric representations of $a(q^r)$, $b(q^r)$, $c(q^r)$ for $r\in 1,2,4,6$, as well as $a(-q)$, $b(-q)$, $c(-q)$, expressed in terms of the parameters p and k, and are presented below.*

$$a(-q)=(1-2p-2p^2)k,$$
$$a(q)=(1+4p+p^2)k,$$
$$a(q^2)=(1+p+p^2)k,$$

$$a(q^4)=(1+p-\frac{1}{2}p^2)k,$$
$$a(q^6)=\frac{(p^2+p+1+2^{1/3}((1-p)(2+p)(1+2p))^{2/3})k}{3},$$
$$b(-q)=2^{-\frac{1}{3}}((1-p)(1+2p)^4(2+p))^{\frac{1}{3}}k,$$
$$b(q)=2^{-\frac{1}{3}}((1-p)^4(1+2p)(2+p))^{\frac{1}{3}}k,$$
$$b(q^2)=2^{-2/3}((1-p)(1+2p)(2+p))^{\frac{2}{3}}k,$$
$$b(q^4)=2^{-\frac{4}{3}}((1-p)(1+2p)(2+p)^4)^{\frac{1}{3}}k,$$
$$c(-q)=-2^{\frac{1}{3}}3(p(1+p))^{\frac{1}{3}}k,$$
$$c(q)=2^{-\frac{1}{3}}3(p(1+p)^4)^{\frac{1}{3}}k,$$
$$c(q^2)=2^{-\frac{2}{3}}3(p(1+p))^{\frac{2}{3}}k. \qquad ,$$
$$c(q^4)=2^{-\frac{4}{3}}3(p^4(1+p))^{\frac{1}{3}}k,$$
$$c(q^6)=\frac{\left(p^2+p+1-2^{-2/3}((1-p)(2+p)(1+2p))^{2/3}\right)k}{3}.$$

3. New Insights into Eisenstein Series and Cubic Theta Functions Inspired by Ramanujan

In Ramanujan's notebook [7], he recorded intriguing series involving variables L, M and N which highlighted important identities for infinite series incorporating theta functions. Building on his work, Xia et al. [10] used computational techniques to uncover elegant identities involving Eisenstein series and cubic theta functions, particularly expressions of the form $L(q)-rL(q^r)$, with $r\in\{2,3,4,6,12\}$. More recently, Vidya H. C. and Ashwath Rao B. [6], along with Vidya H. C. and Smitha G. Bhat [9], have formulated few identities that includes $L(-q^l)-L(q^l)$, where $l\in\{1,3\}$. Vidya H. C. et al. [5, 8], also established important connections between theta functions, advancing the study of these mathematical structures.

Motivated by these works, we have established certain new significant identities that relates Ramanujan type Eisenstein series and cubic theta functions. In this paper, we present new, precise relationships between Ramanujan-type Eisenstein series and cubic theta functions, focusing on Eisenstein series of the form $N(q^n)$, where $n \in \{1, 2, 3, 6\}$. Notably, our approach is entirely analytical. Our findings offer new insights into the intricate connections between these mathematical objects, enhancing the understanding of their underlying relationships in a rigorous framework.

Theorem 3.1. *The relationship between the infinite series and theta functions can be described as follows:*

$$(i)\ \left[-\frac{67}{46}+\frac{9u}{4}\right]+540\sum_{r=1}^{\infty}\left[\left(-\frac{9u}{4}+\frac{37}{322}\right)\frac{r^5q^r}{1-q^r}-\left(\frac{198}{161}\right)\frac{r^5q^{3r}}{1-q^{3r}}\right.$$
$$+\left(\frac{2556}{161}\right)\frac{r^5q^{6r}}{1-q^{6r}}-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3-\left(\frac{25}{138}+\frac{99u}{4}\right)$$
$$\left(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{1}{138}\left(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.\right.$$
$$\left.\left.-\frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3+\left(\frac{1}{8}\right)\left(-2+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right)^3+u\left(-3+24\right.$$
$$\left.\left.\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{47}{138}\left(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\right)^3\right]$$
$$=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.1}$$

$$(ii)\ \left[-\frac{9}{32}+\frac{9u}{4}\right]+540\sum_{r=1}^{\infty}\left(-\frac{9u}{4}\right)\frac{r^5q^r}{1-q^r}-\left(\frac{9}{224}\right)\frac{r^5q^{3r}}{1-q^{3r}}+\left(\frac{9}{28}\right)\frac{r^5q^{6r}}{1-q^{6r}}$$
$$-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3+\left(\frac{1}{96}-\frac{99u}{4}\right)\left(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3+\frac{1}{96}\left(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3+u\left(-3+24\right.$$
$$\left.\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{1}{96}\left(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\right)^3$$

$$= \left\{ a(q)a(q^2) \right\}^3. \tag{3.2}$$

(*iii*)
$$\left[-\frac{33}{46}+\frac{9u}{4}\right]+540\sum_{r=1}^{\infty}\left(\frac{48}{161}-\frac{9u}{4}\right)\frac{r^5q^r}{1-q^r}+\left(\frac{5535}{322}\right)\frac{r^5q^{3r}}{1-q^{3r}} -\left(\frac{2700}{161}\right)\frac{r^5q^{6r}}{1-q^{6r}}-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3+\left(\frac{163}{46}-\frac{99u}{4}\right) \left(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\left(\frac{229}{46}\right)\left(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}} -\frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3-\frac{3}{8}\left(-2+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right)^3 +u\left(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\right)^3+\frac{1}{46}\left(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}} -\frac{rq^r}{1-q^r}\right]\right)^3=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3. \tag{3.3}$$

Proof. Let us persume that,

$$C_1N(q)+C_2N(q^3)+C_3N(q^6)+C_4[L(-q)-L(q)]^3+C_5[L(q)-2L(q^2)]^3 +C_6[2L(q^2)-3L(q^3)]^3+C_7[L(q)-3L(q^3)]^3+C_8[3L(q^3)-4L(q^4)]^3+C_9[L(q) -4L(q^4)]^3+C_{10}[4L(q^4)-6L(q^6)]^3+C_{11}[L(q)-6L(q^6)]^3+C_{12}[L(q) -12L(q^{12})]^3=\left\{a(q)a(q^2)\right\}^3. \tag{3.4}$$

The above equation is transformed by (p,k) parametrization using Lemma 2.3 and solved for unknowns.

$$\begin{pmatrix}
1 & 1 & 1 & 0 & -1 & -8 & -27 & -125 & -1331 & -1 & -1 & -8 \\
6 & 6 & 6 & 0 & -42 & -192 & -486 & -1650 & -12342 & -6 & -6 & -48 \\
-114 & 12 & 12 & 0 & -660 & -1968 & -3888 & -9960 & -51216 & -48 & -12 & -96 \\
-625 & -58 & 5 & 13824 & -4802 & -11200 & -18144 & -36058 & -125896 & -158 & -32 & -40 \\
-\frac{4059}{2} & -297 & -\frac{27}{2} & 62208 & -17019 & -38520 & -54432 & -86607 & -204246 & -603 & -90 & 36 \\
-4302 & -396 & -18 & 124416 & -34524 & -81792 & -108864 & -144540 & -230184 & -1044 & -72 & -144 \\
-5556 & -264 & -12 & 145152 & -43368 & -105888 & -145152 & -171120 & -184296 & -2112 & -168 & -264 \\
-4302 & -396 & -18 & 108864 & -34524 & -81792 & -124416 & -144540 & -105120 & -1044 & -288 & 72 \\
-\frac{4059}{2} & -297 & -\frac{27}{2} & 54432 & -17019 & -38520 & -62208 & -86607 & -42084 & -603 & 180 & 90 \\
-625 & -58 & 5 & 18144 & -4802 & -11200 & -13824 & -36058 & -11416 & -158 & -536 & -292 \\
-114 & 12 & 12 & 3888 & -660 & -1968 & 0 & -9960 & -1968 & -48 & 384 & -60 \\
6 & 6 & 6 & 486 & -42 & -192 & 0 & -1650 & -192 & -6 & -96 & 150 \\
1 & 1 & 1 & 27 & -1 & -8 & 0 & -125 & -8 & -1 & 8 & -125
\end{pmatrix}
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 1 \\ 15 \\ 93 \\ 320 \\ 711 \\ 1125 \\ 1302 \\ 1125 \\ 711 \\ 320 \\ 93 \\ 15 \\ 1 \end{pmatrix}.$$

$$C_1 = \left\{\frac{9u}{4}\right\},\ C_2 = \left\{\frac{9}{224}\right\},\ C_3 = -\frac{9}{28},\ C_4 = -u,\ C_5 = \left\{\frac{1}{96} - \frac{99u}{4}\right\},\ C_6 = \frac{1}{96},$$
$$C_7 = 0,\ C_8 = 0,\ C_9 = u,\ C_{10} = 0,\ C_{11} = -\frac{1}{96} \text{ and } C_{12} = 0. \text{ where } u, v \in \mathbb{R}.$$

By substituting the previously mentioned statistics into (3.4) and streamlining the process with the help of Definition 2.1, we arrive at equation (3.1). Similarly, using a parallel methodology, we derive the ensuing identities. Modifying the right-hand side of (3.1) and then applying (3.4) yields the expressions denoted as equations (i) and (ii).

$$(i)\ \left(-\frac{37}{322} + \frac{9u}{4}\right)N(q) + \frac{198}{161}N(q^3) - \frac{2556}{161}N(q^6) - u[L(-q) - L(q)]^3$$
$$- \left(\frac{25}{138} + \frac{99u}{4}\right)[L(q) - 2L(q^2)]^3 - \frac{1}{138}[2L(q^2) - 3L(q^3)]^3 + \frac{1}{8}[L(q)$$
$$- 3L(q^3)]^3 + u[L(q) - 4L(q^4)]^3 - \frac{47}{138}[L(q) - 6L(q^6)]^3 = \left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3.$$

$$(ii)\ \left(-\frac{48}{161}+\frac{9u}{4}\right)N(q)-\frac{5535}{322}N(q^3)+\frac{2700}{161}N(q^6)-u[L(-q)-L(q)]^3$$
$$-\left(\frac{163}{46}-\frac{99u}{4}\right)[L(q)-2L(q^2)]^3-\frac{229}{46}[2L(q^2)-3L(q^3)]^3$$
$$-\frac{3}{8}[L(q)-3L(q^3)]^3+u[L(q)-4L(q^4)]^3+\frac{1}{46}[L(q)-6L(q^6)]^3$$
$$=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3.$$

Using the Eisenstein series definition for simplification yields equations (3.2) through (3.3). □

Theorem 3.2. *The link between the infinite series and theta functions is established through the following:*

$$(i)\ \left[-\frac{358}{23}+18u\right]+540\sum_{r=1}^{\infty}\left[\left(\frac{148}{161}-18u\right)\frac{r^5q^{2r}}{1-q^{2r}}-\left(\frac{198}{161}\right)\frac{r^5q^{3r}}{1-q^{3r}}\right.$$
$$+\left(\frac{2556}{161}\right)\frac{r^5q^{6r}}{1-q^{6r}}-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3-\left(\frac{68}{69}+9u\right)$$
$$\left(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\right)^3+\frac{1}{8}\left(-2+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3+u\left(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{47}{138}\left(-5\right.$$
$$\left.+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{1}{138}\left(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.\right.$$
$$\left.\left.-\frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.5}$$

$$(ii)\ \left[-\frac{9}{32}+18u\right]+540\sum_{r=1}^{\infty}\left(-18u\right)\frac{r^5q^r}{1-q^r}-\left(\frac{9}{224}\right)\frac{r^5q^{3r}}{1-q^{3r}}+\left(\frac{9}{28}\right)\frac{r^5q^{6r}}{1-q^{6r}}$$
$$-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3+\left(\frac{1}{96}-9u\right)\left(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3+u\left(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\right)^3-\frac{1}{96}\left(-5\right.$$

$$+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^{r}}{1-q^{r}}\right]\Bigg)^{3}+\frac{1}{96}\Bigg(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.$$
$$\left.-\frac{2rq^{2r}}{1-q^{2r}}\right]\Bigg)^{3}=\Big\{a(q)a(q^{2})\Big\}^{3}. \tag{3.6}$$

$$(iii)\ \left[-\frac{129}{46}+18u\right]+540\sum_{r=1}^{\infty}\left(\frac{384}{161}-18u\right)\frac{r^{5}q^{r}}{1-q^{r}}+\left(\frac{5535}{322}\right)\frac{r^{5}q^{3r}}{1-q^{3r}}$$
$$-\left(\frac{2700}{161}\right)\frac{r^{5}q^{6r}}{1-q^{6r}}-u\Bigg(24\sum_{r=1}^{\infty}\frac{rq^{r}}{1-q^{r}}-\frac{r(-q)^{r}}{1-(-q)^{r}}\Bigg)^{3}+\left(\frac{67}{46}-9u\right)$$
$$\Bigg(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^{r}}{1-q^{r}}\right]\Bigg)^{3}-\left(\frac{3}{8}\right)\Bigg(-2+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.$$
$$\left.-\frac{rq^{r}}{1-q^{r}}\right]\Bigg)^{3}+u\Bigg(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^{r}}{1-q^{r}}\right]\Bigg)^{3}+\frac{1}{46}\Bigg(-5$$
$$+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^{r}}{1-q^{r}}\right]\Bigg)^{3}-\frac{229}{46}\Bigg(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}\right.$$
$$\left.-\frac{2rq^{2r}}{1-q^{2r}}\right]\Bigg)^{3}=\left\{\frac{b^{3}(q)b(q^{2})}{b(-q)b(q^{4})}\right\}^{3}. \tag{3.7}$$

Proof. Let us persume that,

$$C_1N(q^2)+C_2N(q^3)+C_3N(q^6)+C_4[L(-q)-L(q)]^3+C_5[L(q)-2L(q^2)]^3$$
$$+C_6[L(q)-3L(q^3)]^3+C_7[L(q)-4L(q^4)]^3+C_8[L(q)-6L(q^6)]^3+C_9[L(q)$$
$$-12L(q^{12})]^3+C_{10}[2L(q^2)-3L(q^3)]^3+C_{11}[3L(q^3)-4L(q^4)]^3+C_{12}[4L(q^4)$$
$$-6L(q^6)]^3=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3. \tag{3.8}$$

By applying the (p, k) parametrization as outlined in Lemma 2.3, the given equation is transformed. This leads to the formulation of a system of non-homogeneous linear equations, where the coefficients of terms involving powers of k^3 and their corresponding powers of p, such as k^3, pk^3, p^2k^3, p^3k^3, and so on up to $p^{12}k^3$, are matched between the left and right hand sides. The next step is to solve these equations to find the unknown values.

$$\left(\begin{matrix}
1 & 1 & 1 & 0 & -1 & -8 \\
6 & 6 & 6 & 0 & -42 & -192 \\
-114 & 12 & 12 & 0 & -660 & -1968 \\
-625 & -58 & 5 & 13824 & -4802 & -11200 \\
-\frac{4059}{2} & -297 & -\frac{27}{2} & 62208 & -17019 & -38520 \\
-4302 & -396 & -18 & 124416 & -34524 & -81792 \\
-5556 & -264 & -12 & 145152 & -43368 & -105888 \\
-4302 & -396 & -18 & 108864 & -34524 & -8179 \\
-\frac{4059}{2} & -297 & -\frac{27}{2} & 54432 & -17019 & -38520 \\
-625 & -58 & 5 & 18144 & -4802 & -11200 \\
-114 & 12 & 12 & 3888 & -660 & -1968 \\
6 & 6 & 6 & 486 & -42 & -192 \\
1 & 1 & 1 & 27 & -1 & -8
\end{matrix}\right.$$

$$\left.\begin{matrix}
-27 & -125 & -1331 & -1 & -1 & -8 \\
-486 & -1650 & -12342 & -6 & -6 & -48 \\
-3888 & -9960 & -51216 & -48 & -12 & -96 \\
-18144 & -36058 & -125896 & -158 & -32 & -40 \\
-54432 & -86607 & -204246 & -603 & -90 & 36 \\
-108864 & -144540 & -230184 & -1044 & -72 & -144 \\
-145152 & -171120 & -184296 & -2112 & -168 & -264 \\
-124416 & -144540 & -105120 & -1044 & -288 & 72 \\
-62208 & -86607 & -42084 & -603 & 180 & 90 \\
-13824 & -36058 & -11416 & -158 & -536 & -292 \\
0 & -9960 & -1968 & -48 & 384 & -60 \\
0 & -1650 & -192 & -6 & -96 & 150 \\
0 & -125 & -8 & -1 & 8 & -125
\end{matrix}\right)
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 1 \\ -12 \\ 66 \\ -220 \\ 495 \\ -792 \\ 924 \\ -792 \\ 495 \\ -220 \\ 66 \\ -12 \\ 1 \end{pmatrix}$$

On solving the above system, we get,

$$C_1 = \left\{\frac{384}{161} + 18u\right\},\ C_2 = -\left\{\frac{5535}{322}\right\},\ C_3 = \frac{2700}{161},\ C_4 = -u,\ C_5 = \left\{\frac{67}{46} - 9u\right\},$$
$$C_6 = -\frac{3}{8}, C_7 = u,\ C_8 = \frac{1}{46},\ C_9 = 0,\ C_{10} = -\frac{229}{46},\ C_{11} = 0 \text{ and } C_{12} = 0.$$

Substituting the previously discussed statistics into (3.8) and simplifying the resulting expression using Definition 2.1, we obtain equation (3.5). Following a similar approach, we derive the corresponding identities. By adjusting the right-hand side of (3.5) and then applying (3.8), we arrive at the expressions labeled as equations (i) and (ii).

$$(i)\ \ -\left(\frac{148}{161} - 18u\right)N(q^2) + \frac{198}{161}N(q^3) - \frac{2556}{161}N(q^6) - u[L(-q) - L(q)]^3 - \left(\frac{68}{69}\right.$$
$$\left. + 9u\right)[L(q) - 2L(q^2)]^3 + \frac{1}{8}[L(q) - 3L(q^3)]^3 + u[L(q) - 4L(q^4)]^3 - \frac{47}{138}[L(q)$$

$$- 6L(q^6)]^3 - \frac{1}{138}[2L(q^2) - 3L(q^3)]^3 = \left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3.$$

$$(ii)\ 18uN(q^2) + \frac{9}{224}N(q^3) - \frac{9}{28}N(q^6) - u[L(-q) - L(q)]^3 + \left(\frac{1}{96} - 9u\right)[L(q)$$
$$- 2L(q^2)]^3 + u[L(q) - 4L(q^4)]^3 - \frac{1}{96}[L(q) - 6L(q^6)]^3 + \frac{1}{96}[2L(q^2) - 3L(q^3)]^3$$
$$= \left\{a(q)a(q^2)\right\}^3.$$

Applying the Eisenstein series definition in the simplification process yields equations (3.6) through (3.7). □

Theorem 3.3. *The correspondence between the infinite series and theta functions is outlined as:*

$$(i)\ \left[-\frac{1006}{69} + u\right] + 540\sum_{r=1}^{\infty}\left[\left(\frac{68}{483} + \frac{u}{7}\right)\frac{r^5q^r}{1-q^r} - \left(\frac{100}{483} + \frac{8u}{7}\right)\frac{r^5q^{2r}}{1-q^{2r}}\right.$$
$$-\left(\frac{198}{161}\right)\frac{r^5q^{3r}}{1-q^{3r}} + \frac{2556}{161}\frac{r^5q^{6r}}{1-q^{6r}} + u\left(-1 + 24\sum_{r=1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}} - \frac{rq^r}{1-q^r}\right)^3$$
$$-\left(\frac{1}{8}\right)\left(-2 + 24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}} - \frac{rq^r}{1-q^r}\right]\right)^3 - \frac{47}{138}\left(-5 + 24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3 - \frac{1}{138}\left(-1 + 24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}} - \frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3 = \left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.9}$$

$$(ii)\ \left[-\frac{7}{24} + u\right] + 540\sum_{r=1}^{\infty}\left(-\frac{1}{672} + \frac{u}{7}\right)\frac{r^5q^r}{1-q^r} - \left(\frac{1}{84} - \frac{8u}{7}\right)\frac{r^5q^{2r}}{1-q^{2r}}$$
$$-\left(\frac{9}{224}\right)\frac{r^5q^{3r}}{1-q^{3r}} + \left(\frac{9}{28}\right)\frac{r^5q^{6r}}{1-q^{6r}} + u\left(-1 + 24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}} - \frac{rq^r}{1-q^r}\right]\right)^3$$
$$+u\left(-3 + 24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}} - \frac{rq^r}{1-q^r}\right]\right)^3 - \frac{1}{96}\left(-5 + 24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3 + \frac{1}{96}\left(-1 + 24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}} - \frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3$$

$$= \Big\{a(q)a(q^2)\Big\}^3. \tag{3.10}$$

$$(iii)\ \Big[-\frac{98}{23}+u\Big]+540\sum_{r=1}^{\infty}\Big(-\frac{67}{322}+\frac{u}{7}\Big)\frac{r^5q^r}{1-q^r}+\Big(\frac{652}{161}-\frac{8u}{7}\Big)\frac{r^5q^{2r}}{1-q^{2r}}$$
$$+\Big(\frac{5535}{322}\Big)\frac{r^5q^{3r}}{1-q^{3r}}-\Big(\frac{2700}{161}\Big)\frac{r^5q^{6r}}{1-q^{6r}}+u\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}$$
$$-\frac{rq^r}{1-q^r}\Big]\Big)^3-\Big(\frac{3}{8}\Big)\Big(-2+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3+\frac{1}{46}\Big(-5$$
$$+24\sum_{r=1}^{\infty}\Big[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{229}{46}\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}$$
$$-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big)^3=\Big\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\Big\}^3. \tag{3.11}$$

Proof. Let us persume that,

$$C_1N(q)+C_2N(q^2)+C_3N(q^3)+C_4N(q^6)+C_5[L(q)-2L(q^2)]^3+C_6[L(q)$$
$$-3L(q^3)]^3+C_7[L(q)-4L(q^4)]^3+C_8[L(q)-6L(q^6)]^3+C_9[L(q)-12L(q^{12})]^3$$
$$+C_{10}[2L(q^2)-3L(q^3)]^3+C_{11}[3L(q^3)-4L(q^4)]^3+C_{12}[4L(q^4)-6L(q^6)]^3$$
$$=\Big\{\frac{a(q)c^2(q)}{c(q^2)}\Big\}^3. \tag{3.12}$$

Applying the (p, k) parametrization as outlined in Lemma 2.3, the given equation is transformed into a system of non-homogeneous linear equations. In this system, the coefficients of terms involving k^3, pk^3, p^2k^3, p^3k^3, p^4k^3, p^5k^3, p^7k^3, p^8k^3, p^9k^3, $p^{10}k^3$, $p^{11}k^3$, and $p^{12}k^3$ on the left-hand side are matched with their corresponding counterparts on the right-hand side. The next step is to solve these equations to determine the unknowns..

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & -1 & -8 & -27 & -125 & -1331 & -1 & -1 & -8 \\
-246 & 6 & 6 & 6 & -42 & -192 & -486 & -1650 & -12342 & -6 & -6 & -48 \\
-5532 & -114 & 12 & 12 & -660 & -1968 & -3888 & -9960 & -51216 & -48 & -12 & -96 \\
-38614 & -625 & -58 & 5 & -4802 & -11200 & -18144 & -36058 & -125896 & -158 & -32 & -40 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & -17019 & -38520 & -54432 & -86607 & -204246 & -603 & -90 & 36 \\
-276084 & -4302 & -396 & -18 & -34524 & -81792 & -108864 & -144540 & -230184 & -1044 & -72 & -144 \\
-348024 & -5556 & -264 & -12 & -43368 & -105888 & -145152 & -171120 & -184296 & -2112 & -168 & -264 \\
-276084 & -4302 & -396 & -18 & -34524 & -81792 & -124416 & -144540 & -105120 & -1044 & -288 & 72 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & -17019 & -38520 & -62208 & -86607 & -42084 & -603 & 180 & 90 \\
-38614 & -625 & -58 & 5 & -4802 & -11200 & -13824 & -36058 & -11416 & -158 & -536 & -292 \\
-5532 & -114 & 12 & 12 & -660 & -1968 & 0 & -9960 & -1968 & -48 & 384 & -60 \\
-246 & 6 & 6 & 6 & -42 & -192 & 0 & -1650 & -192 & -6 & -96 & 150 \\
1 & 1 & 1 & 1 & -1 & -8 & 0 & -125 & -8 & -1 & 8 & -125
\end{pmatrix}
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 27 \\ 486 \\ 3726 \\ 16038 \\ 43173 \\ 76788 \\ 92772 \\ 76788 \\ 43173 \\ 16038 \\ 3726 \\ 486 \\ 27 \end{pmatrix}
$$

On solving the above system, we get,

$$
C_1 = \left\{ -\frac{68}{483} + \frac{u}{7} \right\},\ C_2 = \left\{ \frac{100}{483} + \frac{8u}{7} \right\},\ C_3 = \frac{198}{161},\ C_4 = -\frac{2556}{161},\ C_5 = u,
$$
$$
C_6 = \frac{1}{8},\ C_7 = 0,\ C_8 = -\frac{47}{138},\ C_9 = 0,\ C_{10} = -\frac{1}{138},\ C_{11} = 0
$$
$$
\text{and } C_{12} = 0. \text{ where } u, v \in \mathbb{R}.
$$

By substituting the previously mentioned statistics into (3.12) and simplifying the resulting expression using Definition 2.1, we obtain equation (3.10). Following a similar approach, the subsequent identities are derived. Adjusting the right hand side of (3.10) and then utilizing (3.12) leads to the expressions labeled as equations

(i) and (ii).

$$(i)\ \left(\frac{1}{672}-\frac{u}{7}\right)N(q)-\left(\frac{1}{84}-\frac{8u}{7}\right)N(q^2)+\frac{9}{224}N(q^3)-\frac{9}{28}N(q^6)+u[L(q)$$
$$-2L(q^2)]^3+-\frac{1}{96}[L(q)-6L(q^6)]^3++\frac{1}{96}[2L(q^2)-3L(q^3)]^3=\left\{a(q)a(q^2)\right\}^3$$

$$(ii)\ \left(\frac{67}{322}-\frac{u}{7}\right)N(q)-\left(\frac{652}{161}-\frac{8u}{7}\right)N(q^2)-\frac{5535}{322}N(q^3)+\frac{2700}{161}N(q^6)+u[L(q)$$
$$-2L(q^2)]^3-\frac{3}{8}[L(q)-3L(q^3)]^3+\frac{1}{46}[L(q)-6L(q^6)]^3-\frac{229}{46}[2L(q^2)$$
$$-3L(q^3)]^3=\left\{\frac{b^3(q)b(^2)}{b(-q)b(q^4)}\right\}^3.$$

Simplifying using the Eisenstein series definition leads us to equations (3.10) through (3.11). □

4. Conclusion

The convergence of the Eisenstein series to infinite product underscores their universality and versatility in mathematics. The product representation offers computational advantages, enabling efficient evaluation of these functions for specific arguments and serves as a conceptual bridge between discrete summations and continuous transformations.

In symmetry, the link between Eisenstein series and infinite products not only simplifies their analytic study but also enhances our understanding of their modular properties and broader the connections with the landscape of special functions and modular forms. Moreover, the infinite product can reveal modular relation between Eisenstein series and other modular forms, such as theta functions. This interconnection enriches the study of modular symmetries and enables explicit computations in area like partition theory, q-series and string theory.

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