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LIE ALGEBRA BUNDLES DEFINED BY JORDAN ALGEBRA BUNDLES OF FINITE TYPE

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ABSTRACT. In this paper, we discuss the Lie algebra bundles defined by Jordan algebra bundles of finite type over an arbitrary topological space and observe that such Lie algebra bundles are also of finite type. Also, we examine some ideal bundles of Lie algebra bundles defined by semisimple Jordan algebra bundles of finite type.

1. Introduction

Let F denotes either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. For a topological space X, let $C(X) = C_F(X) = F^X$ denote the ring of continuous F-valued functions on X.

Serre (1955) [19] has established that there is a one-to-one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its co-ordinated ring. Later, Swan (1961) [20] has established that if X is a compact Hausdorff space, the category of complex vector bundles over X is equivalent to the category of finitely generated projective $C(X)$-modules which is known as Serre-Swan Correspondence. Vaserstein (1986) [21] introduced the notion of vector bundles of finite type over an arbitrary topological space and proved that the category of finitely generated projective F^X-modules is equivalent to the category of F-vector bundles of finite type over an arbitrary space X. Ranjitha Kumar et al. [11] have studied Lie algebra bundles of finite type over an arbitrary space X and proved that there is a bijection between Lie algebra bundles of finite type over a topological space X and finitely generated projective Lie rings over the ring of continuous functions on the base space X.

Motivated by the works of Koecher [9, 10], G. Prema and B. S. Kiranagi [17] have given a construction of a Lie algebra bundle by Jordan algebra bundle over a compact Hausdorff space. In this paper, we construct Lie algebra bundles through Jordan algebra bundles of finite type over an arbitrary space X. We observe that the constructed Lie algebra bundles are also bundles of finite type. When the Jordan bundle is semisimple, we see that some ideal bundles of the constructed Lie algebra bundles are also of finite type. For basic terminologies in vector bundles

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we follow the textbooks [3] and [5]. Some recent developments in the theory of algebra bundles can be found in [1,2,12–16]. Throughout this paper, it is assumed that all underlying vector spaces are real and finite dimensional.

2. Definitions

We recall the following basic definitions which can be found in [4,6–8,11,17,18].

Definition 2.1. A Jordan(Lie) algebra bundle for short a Jordan(Lie) bundle, is a vector bundle ξ over a topological space X, together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces a Jordan (Lie) algebra structure on each fibre ξ.

Definition 2.2. A morphism $\phi : \xi \to \eta$ of Jordan (Lie) bundles ξ, η over X is a morphism of the underlying vector bundles such that for every x in X, $\phi_x : \xi_x \to \eta_x$ is a Jordan (Lie) algebra homomorphism. A morphism ϕ is an isomorphism if it is bijective and ϕ^{-1} is continuous.

Definition 2.3. A locally trivial Jordan (Lie) algebra bundle is a vector bundle ξ in which each fibre is a Jordan (Lie) algebra and for each x in X there is an open set U in X containing x, a Jordan (Lie) algebra L and a homeomorphism $\phi : U \times L \to p^{-1}(U)$ such that for each x in U, $\phi_x : \{x\} \times L \to p^{-1}(x)$ is a Jordan (Lie) algebra isomorphism.

Definition 2.4. A Jordan (Lie) bundle $\eta = (\eta, q, X)$ is a subbundle of a Jordan (Lie) bundle $\xi = (\xi, p, X)$ provided η is a vector subbundle of ξ and for each x in X, η_x is a subalgebra of the Jordan (Lie) algebra ξ_x. If η is a subbundle of ξ such that each η_x is an ideal in ξ_x, then η is called an ideal bundle of ξ.

Definition 2.5. A section of a Jordan (Lie) algebra bundle (ξ, π, X) is a map $s : X \to \xi$ such that $p \circ s = id_X$.

The symbol $\Gamma(\xi)$ denotes the set of all sections of ξ.

Definition 2.6. A semisimple Jordan (Lie) algebra bundle is a vector bundle ξ in which the morphism $\theta : \xi \oplus \xi \to \xi$ induces a semisimple Lie algebra structure on each fibre ξ_x.

Definition 2.7. A Jordan (Lie) algebra bundle ξ over an arbitrary space X is of finite type if there is a finite partition S of unity on X (that is a finite set S of non-negative continuous functions on X whose sum is 1) such that the restriction of the bundle to the set $\{x \in X \mid f(x) \neq 0\}$ is a trivial Jordan (Lie) algebra bundle for each f in S.

Example: Any Jordan (Lie) algebra bundle over a compact Hausdorff space is of finite type.

3. Construction and Results

We shall discuss the construction of a Lie algebra bundle of finite type from a Jordan algebra bundle of finite type in the following Lemma 3.1 and Theorem 3.2.

Lemma 3.1. *Let ξ be a Jordan algebra bundle of finite type over an arbitrary topological space X, with each fibre ξ_x having a unit element e_x, $x \in X$. Then $g(\xi) = \bigcup_{x\in X} g(\xi_x) = \bigcup_{x\in X} (Der(\xi_x) + L(\xi_x))$ is a Lie algebra bundle of finite type.*

Proof. Since ξ is a Lie algebra bundle of finite type, there exists a finite partition of unity $\{f_i\}_{i=1}^n$ such that $\xi|_{U_i}$ is trivial, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Let $\phi_i : U_i \times J_i \to p^{-1}(U_i) = \xi|_{U_i}$ be the the isomorphism, where J_i is a Jordan algebra. Then $\phi_i(x, ab) = \theta_x(\phi(x,a), \phi(x,b))$. Each $g(\xi_x)$ is a Lie algebra, where the multiplication $g(\theta_x) : g(\xi_x) \oplus g(\xi_x) \to g(\xi_x)$ is given by

$$g(\theta_x(D_i + La, D_i' + La')) = [D_i, D_i] \oplus ([La, La'] + LD_i a' - LD_i' a)\,,$$

for all D_i', D_i in $Der(\xi_x)$ and a', a in ξ_x.

Let $g(\phi_i) = End\,\phi_i|_{U_i \times g(J_i)}$, where the vector bundle morphism, $End\,\phi_i : U_i \times End\,J_i \to \bigcup_{x \in X} End\,\xi_x$ is given by $End\,\phi_i(x,f) = \phi_{i_x} f \phi_{i_x}^{-1}$, in which ϕ_{i_x} is the isomorphism $\phi_i|_x$, ϕ_i restricted to x. Consider $D_i + La \in g(J_i)$. Then $g(\phi_i)(D_i + La) = \phi_{i_x}(D_i + La)\phi_{i_x}^{-1}$. Now, $D_i' = \phi_x D_i \phi_{i_x}^{-1} \in Der\xi_x$ because for all $s, t \in \xi_x$,

$$\begin{aligned}
D_i'(\theta_x(s,t)) &= (\phi_{i_x} D_i \phi_{i_x}^{-1})\theta_x(s,t) \\
&= \phi_{i_x} D_i \phi_{i_x}^{-1}(s) \phi_{i_x}^{-1}(t) \\
&= \phi_i(\phi_{i_x}^{-1}(s) D_i \phi_{i_x}^{-1}(t)) - D_i(\phi_{I_x}^{-1}(s))\phi_{i_x}^{-1}(t) \\
&= \theta_x(s, \phi_{i_x} D_i \phi_{i_x}^{-1}(t)) - \theta_x(\phi_{i_x} D_i \phi_{i_x}^{-1}(s), t) \\
&= \theta_x(s, D_i'(t)) - \theta_x(D_i'(s), t).
\end{aligned}$$

For $s \in \xi_x$, $\phi_{i_x} La \phi_{i_x}^{-1}(s) = \phi_{i_x} a \phi_{i_x}^{-1}(s) = \theta_x(\phi_{i_x}(a), s) = L(\phi_{i_x}(a))(x)$. Then $\phi_{i_x} La \phi_{i_x}^{-1} = L(\phi_{i_x}(a)) \in L(\xi_x)$. Thus $g(\phi_i)$ maps $U_i \times g(J_i)$ onto $\bigcup_{x \in U_i} g(\xi_x)$, and it is an isomorphism. It remains to verify that $g(\phi_i)$ restricted to each fibre is a Lie algebra isomorphism. Let $D_i + La$, $D_i' + La' \in g(J_i)$. Then

$$\begin{aligned}
&g(\phi_i)(x, [D_i + La, D_i' + La']) \\
&\quad= g(\phi_i)(s, [D_i, D_i'] + La, La' + LD_i(a') - LD_i'(a)) \\
&\quad= \phi_{i_x}[D_i, D_i']\phi_{i_x}^{-1} + \phi_{i_x}[L(a), L(a')]\phi_{i_x}^{-1} + \phi_{I_x}(L(D_i)a')\phi_{i_x}^{-1} \\
&\qquad\qquad - \phi_{i_x} L(D_i'(a))\phi_{i_x}^{-1} \\
&\quad= [\phi_{i_x} D_i \phi_{i_x}^{-1}, \phi_{i_x} D_i' \phi_{i_x}^{-1}] + [L(\phi_{i_x}(a)), L(\phi_{i_x}(a'))] \\
&\qquad\qquad + L(\phi_{i_x}(D_i \phi_{i_x}^{-1}(\phi_{i_x}(a))) - L(\phi_{i_x} D_i' \phi_{i_x}^{-1}(\phi_{i_x}(a))) \\
&\quad= g(\theta)(g(\phi_i)(D_i + La), g(\phi_i)(D_i + La')).
\end{aligned}$$

Hence $g(\xi) = (g(\xi), g(p), X, g(\theta))$ is a Lie algebra bundle of finite type, where $g(p) : g(\xi) \to X$ is defined by $g(p)(T) = x$ if $T \in g(\xi_x)$ is an isomorphism. □

Theorem 3.2. *If ξ is a Jordan bundle of finite type in which each fibre has a unit element, then $K(\xi) = g(\xi) \oplus \xi \oplus \bar{\xi}$ is a Lie algebra bundle of finite type, where $\bar{\xi}$ is an isomorphic copy of the Jordan bundle ξ.*

Proof. Since ξ is a Lie algebra bundle of finite type, there exists a finite partition of unity $\{f_i\}_{i=1}^n$ such that $\xi|_{U_i}$ is trivial, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Let $\phi_i : U_i \times J_i \to p^{-1}(U_i) = \xi|_{U_i}$ be the isomorphism, where J_i is a Jordan algebra. If $\bar{J}_i$ is an isomorphic copy of the Jordan algebra J_i, we can define

$$\bar{\phi_i} : U_i \times \bar{J_i} \to \bigcup_{x \in U_i} \bar{\xi_x} \quad \text{by} \quad \phi_i(s, \bar{a}) = \overline{\phi_i(s,a)}.$$

Then $\bar{\phi_i}$ is an isomorphism and hence $\bar{\xi}$ is of finite type. Now $K(\xi)$ is a vector bundle being the direct sum of the vector bundles $g(\xi)$, ξ and $\bar{\xi}$. Each fibre $K(\xi_x)$ is a Lie algebra with Lie multiplication

$$K(\theta_x) : K(\xi_x) \oplus K(\xi_x) \to K(\xi_x)$$

given by,

$$\begin{aligned} &K(\theta_x)(T_1 + a_1 + \bar{b_1}, T_2 + a_2 + \bar{b_2}) \\ &\quad = g(\theta_x)(T_1, T_2) + a_1 \circ b_2 - a_2 \circ b_1 \oplus T_1 a_2 - T_2 a_1 \oplus \{\overline{T_2^* b_1 - T_1^* b_2}\}, \end{aligned}$$

for $T_1, T_2 \in g(\xi_x)$ and $a_1, a_2, b_1, b_2 \in \xi_x$, where $T^* = -D + La$ for $T = D + La$ and $a \circ b = 2\{L(ab) + [L(a), L(b)]\}$. The morphism

$$K(\phi_i) = g(\phi_i) \oplus \phi_i \oplus \bar{\phi_i} : U_i \times K(J_i) \to \bigcup_{x \in U_i} K(\xi_x)$$

is a vector bundle morphism. We note that

$$(\phi_i)_x(Ta) = (\phi_i)_x T (\phi_i)_x^{-1} (\phi_i)_x(a) = (g(\phi_i))_x(T)(\phi_i)_x(a))$$

and

$$(\phi_i)_x(T^*(a)) = ((g\phi_i))_x T)^*((\phi_i)_x(a)).$$

Now

$$\begin{aligned} &(K(\phi_i))_x)[T_1 \oplus a_1 \oplus \bar{b_1}, T_2 \oplus a_2 \oplus \bar{b_2}] \\ &\qquad = [(K(\phi_i)_y)(T_1 \oplus a_1 \oplus \bar{b_1}), K(\phi_i)_y(T_2 \oplus a_2 \oplus \bar{b_2})]. \end{aligned}$$

Thus, $K(\phi_i)|_{y \times K(J_i)}$ is a Lie algebra isomorphism. Hence $K(\xi)$ is a Lie algebra bundle of finite type and the morphism $K(\theta) : K(\xi) \oplus K(\xi) \to K(\xi)$ is given by

$$\begin{aligned} &K(\theta)(T_1 \oplus a_1 \oplus \bar{b_1}, T_2 \oplus a_2 \oplus \bar{b_2}) \\ &\quad = \{[T_1, T_2] + 2(La_1 b_2 + [La_1, Lb_2]) \\ &\qquad - 2(La_2 b_1 + [La_2, Lb_1])\} \oplus \{T_1 a_2 - T_2 a_1\} + \{\overline{T_2^* b_1 - T_1^* b_2}\}, \end{aligned}$$

where T_1, T_2 are in $g(\xi_x)$, a_1, a_2, b_1, b_2 are in ξ_x and $T^* = -D + La$ whenever $T = D + La$, gives the Lie bundle structure on $K(\xi)$. □

Lemma 3.3. *Let ξ be a Jordan algebra bundle of finite type. Consider $h(\xi) = \bigcup_{x \in X} h(\xi_x)$, where $h(\xi_x) = L(\xi_x) \oplus [L(\xi_x), L(\xi_x)]$. Then $\mathfrak{L}(\xi) = h(\xi) \oplus \xi \oplus \xi$ is an ideal bundle of $K(\xi)$ and it is of finite type.*

Proof. We observe that if ξ is of finite type, then $h(\xi)$ is also a Lie algebra bundle of finite type. For, let $\{f_i\}_{i=1}^n$ be a partition of unity such that $\xi|_{U_i}$ is trivial, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Let $\phi_i : U_i \times J_i \to \bigcup_{x \in U_i} \xi_x$ be the isomorphisms. Then $h(\phi_i) = g(\phi_i)|_{U_i \times h(J_i)} : U_i \times h(J_i) \to \bigcup_{x \in U_i} h(\xi_x)$ is an isomorphism and hence $h(\xi)$ is of finite type. Thus, $\mathfrak{L}(\xi)$ is a bundle of finite type, by the above theorem. Since each $\mathfrak{L}(\xi_x)$ is an ideal of $K(\xi_x)$, $\mathfrak{L}(\xi)$ is an ideal bundle of $K(\xi)$. □

The following lemma is straightforward and hence it is stated without proof.

Lemma 3.4. *Let $(X \times V, p, X)$ be a trivial vector bundle and $(X \times J, p_J, X)$ be a trivial semisimple Jordan bundle. Suppose $\phi : X \times V \to X \times J$ is a vector bundle monomorphism such that for each x in X, $\phi_x(V)$ is an ideal in J. Then there exists a finite open partition $\bigcup\limits_i X_i = X$, such that $\phi_x(V) = \phi_y(V)$ for all x, y in X_i.*

Proposition 3.5. *Let η be an ideal bundle of finite type of a semi simple Jordan bundle of finite type ξ. Then $h_1(\eta) = \bigcup\limits_{x\in X} h_1(\eta_x)$ is an ideal bundle of $h(\xi)$, where $h_1(\eta_x) = \{T \in h(\xi_x) \mid T(\xi_x) \subset \eta_x\}$ is an ideal in $h(\xi_x)$. Moreover, $h_1(\eta)$ is an ideal bundle of finite type.*

Proof. Since ξ is of finite type, there exists a finite partition $\{f_i\}_{i=1}^n$ of unity such that $\eta|_{U_i}$ is trivial, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$.

Let $\phi_i : U_i \times J_i \to \bigcup\limits_{x\in U_i} \xi_x$ be the isomorphisms. Since η is an ideal bundle of finite type, we have the isomorphisms, $\psi_i : U_i \times J_i' \to \bigcup\limits_{x\in U_i} \eta_x$, where $J_i' \subseteq J_i$. Then $\phi_i^{-1}\psi_i : U_i \times J_i' \to U_i \times J_i$ is a vector bundle monomorphism satisfying hypothesis of the Lemma 3.4. Hence there exists a finite open partition $\bigcup_j U_{ij} = U_i$ such that $\phi_i^{-1}\psi_i(s, J') = \phi_i^{-1}\psi_i(s', J')$, for all $s, s' \in U_{ij}$. The neighbourhood U_i can be shrinked so that there exists an ideal I_i of J_i such that $\phi_i : U_i \times I_i \to \bigcup\limits_{x\in U_i} \eta_x$ is an isomorphism.

Let $h_1(I_i) = \{T \in h(J_i) \mid T(J_i) \subseteq I_i\}$. Given $T \in h_1(I_i)$, consider $(g(\phi_i))_x(T) = \phi_i)xT\phi_{ix}^{-1}$. We have

$$(g(\phi_i))_x(T)(\xi_x) = \phi_{ix}T\phi_{ix}^{-1}(\xi_x) = (\phi_{ix}T)(J) \subseteq \phi_{ix}(I_i) = \eta_x.$$

Thus $g(\phi_i) : U_i \times h_1(I_i) \to \bigcup\limits_{x\in U_i} h_1(\eta_x)$ is an isomorphism. $\square$

Proposition 3.6. *Let ξ be a semisimple Jordan bundle of finite type and $\beta = \bigcup\limits_{x\in X} \beta_x = \bigcup\limits_{x\in X} (h_x' \oplus A_x \oplus \bar{B}_x)$ be an ideal bundle of the semisimple Lie bundle $\mathfrak{L}(\xi)$, where h_x' is a submodule of $h(\xi_x)$ and A_x, B_x are ideals of ξ_x. Then $\eta = \bigcup\limits_{x\in X} A_x$ and $\eta_1 = \bigcup\limits_{x\in X} B_x$ are ideal bundles of finite type in ξ.*

Proof. Let $\{f_i\}_{i=1}^n$ be a partition of unity such that $\xi|_{U_i}$ is trivial, where $U_i = \{x \in X \mid f_i(x) \neq 0\}$. Let $\phi_i : U_i \times J_i \to \bigcup\limits_{x\in U_i} \xi_x$ be the isomorphisms. Then $\mathfrak{L}(\xi)$ is a Lie algebra bundle of finite type, with the isomorphisms $\mathfrak{L}(\phi_i) = g(\phi_i) \oplus \phi_i \oplus \bar{\phi}_i : U_i \times \mathfrak{L}(J_i) \to \bigcup\limits_{x\in U_i} \mathfrak{L}(\xi_x)$. Since β is an ideal of the semisimple Lie bundle $\mathfrak{L}(\xi)$, we have

$$U_i \times L_i \xrightarrow{\psi_i} \bigcup_{x\in U_i} \beta_x \xrightarrow{i} \bigcup_{x\in U_i} \mathfrak{L}(\xi_x) \xrightarrow{(\mathfrak{L}(\phi_i))^{-1}} U_i \times \mathfrak{L}(J_i),$$

where L_i is a Lie algebra and ψ_i is a Lie bundle morphism. Then by Lemma 3.3, we have an ideal $I = h'(J) \oplus A \oplus \bar{B}$ of $\mathfrak{L}(J_i)$ such that $\mathfrak{L}(\phi_i) : U_i \times I \to \bigcup\limits_{x\in U_i} \beta_x$

is a homeomorphism. Then $\phi : U_i \times A \to \bigcup_{x\in U_i} A_x$ and $\phi_i : U_i \times B \to \bigcup_{x\in U_i} B_y$ give Lie bundle isomorphisms and hence η and η_1 are ideal bundles of finite type. $\square$

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