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A MATHEMATICAL MODEL FOR LINEAR APPROXIMATION OPERATORS CONSTRUCTED USING WAVELETS

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Abstract. It is well known that wavelets are mathematical functions that divide values of functions (signals) into different time and frequency components. Hence they have the great advantage to identify fine details in a signal. The main advantages of Wavelet analysis compared to the Fourier analysis are that they offer simultaneous localization in the frequency and time domain. In this work, we focus on the wavelet type generalized sampling operators. Moreover we will state some of their structural and approximation results.

1. Introduction

Let $B(\mathbb{R}), C(\mathbb{R})$ and $L^p(\mathbb{R})(1 \leq p < \infty)$ are the spaces of bounded, continuous and Lebesgue intagrable functions, respectively, with their well-known usual norms.

The generalized sampling series, which Paul Leo Butzer presented to the world of mathematics and gives the general solution of the approximation problem on the entire real axis, are defined as

$$(S_n f)(t) := \sum_{k\in\mathbb{Z}} f\left(k/n\right)\varphi(nt-k), \qquad (t \in \mathbb{R}), \tag{1.1}$$

where $f \in B(\mathbb{R})$,

$$\varphi \in C(\mathbb{R}) \cap L^1(\mathbb{R}),\ \sum_{k\in\mathbb{Z}} \varphi(q-k) = 1,\ \forall q \in \mathbb{R}, \tag{1.2}$$

and the moments satisfy

$$m_\nu(\varphi) := \sum_{k\in\mathbb{Z}} \varphi(q-k)(q-k)^\nu = C < \infty$$

for $\nu = 1, 2$ and for some $\kappa > 0$

$$M_\kappa(\varphi) := \sup_{q\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi(q-k)|\, |q-k|^\kappa < \infty$$

holds true (see [5]-[6], [8]-[10] and [13]).

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The main focus of this work is to create an operator using Daubechies' compactly-supported wavelets (see [1], [12] and some new papers of the authors [14], [15], [16] and [17]). The new results represent a natural extensions of the results for classical generalized sampling operators and their Kantorovich type modifications.

2. Auxiliary results

A family $(Dj)_{j\in Z}$ of closed sub-spaces of $L^2(\mathbb{R})$ satisfies

v1)

$$... \subset D_{-1} \subset D_0 \subset D_1 \subset ...$$

$$closure\left(\bigcup_j D_j\right) = L_2(\mathbb{R}), \bigcap_j D_j = \{0\}$$

v2)

$$\forall j \in \mathbb{Z},\ \varsigma(\cdot) \in D_j \Leftrightarrow \varsigma(2\cdot) \in D_{j+1},$$

$$\forall j, i \in \mathbb{Z},\ \varsigma(\cdot) \in D_j \Leftrightarrow \varsigma(\cdot - 2^{-j}i) \in D_j,$$

is called Multi-Resolution Analysis (MRA).

Relating to the sets $(Dj)_{j\in Z}$ and MRA, wavelets refer to the set of orthonormal functions of the form

$$\phi_{\eta,\nu}(t) = \frac{1}{\sqrt{\eta}}\phi\left(\frac{t-\nu}{\eta}\right),\ \vartheta_{\eta,\nu}(t) = \frac{1}{\sqrt{\eta}}\vartheta\left(\frac{t-\nu}{\eta}\right), \quad \eta > 0, \nu \in \mathbb{R},$$

where ϕ and ϑ have finite energy and orthogonal, called scale functions (father) and the corresponding wavelets (mother), respectively. The father and mother wavelets satisfy

$$\int_{\mathbb{R}} \phi(\rho)d\rho = 1, \int_{\mathbb{R}} \vartheta(\rho)d\rho = 0.$$

Note that the elements of $(Dj)_{j\in Z}$ are the father wavelets and whereas the elements of their orthogonal complements are the mother wavelets. See [11], [18], [19], [15] and [16].

As an important and useful wavelet is due to I. Daubechies, which construct an orthonormal basis for L_2, given as:

Assume that $\psi \in L_\infty(\mathbb{R})$ satisfies;

w1) supp $(\psi) \subset [0, \lambda]$ $(\lambda > 0)$,
w2) $\int_{\mathbb{R}} \psi(\rho)d\rho = 1$,
w3)

$$\int_{\mathbb{R}} \rho^j \psi(\rho)d\rho = 0, \quad j = 1, ..., N.$$

Inspired by the above informations, we first create an operator through the compactly supported wavelets of Daubechies discussed in this article.

Definition 1. Let $f \in C(\mathbb{R})$, φ is a kernel function and let $\psi \in L_\infty(\mathbb{R})$ satisfies **w1)-w3)**. The wavelet type operators are defined by:

$$\begin{aligned}(WS_n f)(t) \quad : &= n\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_{\mathbb{R}} f(\rho)\,\psi(n\rho-k)d\rho \qquad (2.1)\\ &= \sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_0^{\lambda} f\left(\frac{\rho+k}{n}\right)\psi(\rho)d\rho, \qquad (t\in\mathbb{R}).\end{aligned}$$

Remark 1. Let $\psi(\rho) := \chi_{[0,1]}(\rho)$, then one has

$$\begin{aligned}(WS_n f)(t) &= n\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_{\mathbb{R}} f(\rho)\,\psi(n\rho-k)d\rho\\ &= \sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_0^{1} f\left(\frac{\tau+k}{n}\right)\psi(\tau)d\tau.\end{aligned}$$

This shows that our operators contains the Kantorovich operators as a special case.

3. Fundamental and some convergence properties

We are now ready to establish a strong and interesting relation between the original and wavelet type operators, namely (1.1) and (2.1):

Theorem 1. Let $f \in C(\mathbb{R})$, then

$$(WS_n x^\ell)(t) = (S_n x^\ell)(t), \quad \ell = 0, 1, ..., K$$

holds true.

Proof. We have

$$\begin{aligned}(WS_n x^\ell)(t) &= n\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_{\mathbb{R}} x^\ell\psi(nx-k)dx\\ &= \frac{1}{n^\ell}\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int_{\mathbb{R}}\left[\sum_{i=0}^{\ell}\binom{\ell}{i}\tau^i k^{\ell-i}\right]\psi(\tau)d\tau.\end{aligned}$$

By **w3)**, for $i \neq 0$

$$\int_{\mathbb{R}}\left[\sum_{i=0}^{\ell}\binom{\ell}{i}\tau^i k^{\ell-i}\right]\psi(\tau)d\tau = 0$$

and for $i=0$ we get

$$\begin{aligned}(WS_n x^\ell)(t) &= \frac{1}{n^\ell}\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int\limits_{\mathbb{R}} k^\ell\psi(\tau)d\tau \\ &= \sum_{k\in\mathbb{Z}}\frac{k^\ell}{n^\ell}\varphi(nt-k) \\ &= (S_n x^\ell)(t).\end{aligned}$$

Remark 2. Moreover we obtain

$$\begin{aligned}(WS_n\,(x-t)^\ell)(t) &= n\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int\limits_{\mathbb{R}}(x-t)^\ell\,\psi(nx-k)dx \\ &= \sum_{k\in\mathbb{Z}}\varphi(nt-k)\int\limits_{\mathbb{R}}\left(\frac{\tau+k}{n}-t\right)^\ell\psi(\tau)d\tau \\ &= \frac{1}{n^\ell}\sum_{k\in\mathbb{Z}}\varphi(nt-k)\int\limits_{\mathbb{R}}\left[\sum_{i=0}^{\ell}\binom{\ell}{i}\tau^i\,(nt-k)^{\ell-i}\right]\psi(\tau)d\tau.\end{aligned}$$

Owing to **w2)** and **w3)** we get

$$\begin{aligned}(WS_n\,(x-t)^\ell)(t) &= \frac{1}{n^\ell}\sum_{k\in\mathbb{Z}}\varphi(nt-k)\,(k-nt)^\ell \\ &= (S_n\,(x-t)^\ell)(t).\end{aligned}$$

Theorem 2. At each continuity point ρ_0 of f

$$\lim_{n\to\infty}(WS_n f)(\rho_0)=f(\rho_0)$$

holds true.

Proof. Clearly

$$|f(\rho)-f(\rho_0)|<\epsilon$$

holds true when $|\rho-\rho_0|<\delta$, and

$$|f(\rho)-f(\rho_0)|\leq 2\,\|f\|$$

holds true, when $|\rho-\rho_0|\geq\delta$.

So one has

$$\begin{aligned}(WS_n f)(\rho_0)-f(\rho_0) &= n\sum_{k\in\mathbb{Z}}\varphi(n\rho_0-k)\int\limits_{\mathbb{R}}f(\rho)\,\psi(n\rho-k)d\rho-f(\rho_0) \\ &= \sum_{k\in\mathbb{Z}}\varphi(n\rho_0-k)\int\limits_{\mathbb{R}}f\left(\frac{\tau+k}{n}\right)\psi(\tau)d\tau-f(\rho_0).\end{aligned}$$

We know that

$$(WS_n 1)(t)=(S_n 1)(t)=1. \tag{3.1}$$

Hence

$$\begin{aligned} \left|\left(WS_nf\right)(\rho_0)-f(\rho_0)\right| &= \left|n\sum_{k\in\mathbb{Z}}\varphi(n\rho_0-k)\int\limits_{\mathbb{R}}\left(f\left(\rho\right)-f(\rho_0)\right)\psi(n\rho-k)d\rho\right| \\ &\leq \sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\mathbb{R}}\left|f\left(\frac{\tau+k}{n}\right)-f(\rho_0)\right|\left|\psi(\tau)\right|d\tau \end{aligned}$$

Let us write

$$\left|\left(WS_nf\right)(\rho_0)-f(\rho_0)\right|\leq P_1+P_2,$$

where

$$\begin{aligned} P_1 &= \sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\mathbb{R}}\left|f\left(\frac{\tau+k}{n}\right)-f(\rho_0)\right|\left|\psi(\tau)\right|d\tau \\ &= \sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\left|\frac{\tau+k}{n}-\rho_0\right|<\delta}\left|f\left(\frac{\tau+k}{n}\right)-f(\rho_0)\right|\left|\psi(\tau)\right|d\tau \end{aligned}$$

and

$$P_2=\sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\left|\frac{\tau+k}{n}-\rho_0\right|\geq\delta}\left|f\left(\frac{\tau+k}{n}\right)-f(\rho_0)\right|\left|\psi(\tau)\right|d\tau$$

Hence one has

$$\begin{aligned} P_1 &= \sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\left|\frac{\tau+k}{n}-\rho_0\right|<\delta}\left|f\left(\frac{\tau+k}{n}\right)-f(\rho_0)\right|\left|\psi(\tau)\right|d\tau \\ &\leq M_0\left(\varphi\right)\epsilon\left\|\psi\right\|_\infty, \end{aligned}$$

and

$$\begin{aligned} P_2 &\leq 2\left\|f\right\|\sum_{k\in\mathbb{Z}}\left|\varphi(n\rho_0-k)\right|\int\limits_{\left|\frac{\tau+k}{n}-\rho_0\right|\geq\delta}\left|\psi(\tau)\right|d\tau \\ &\leq 2\left\|f\right\|\frac{M_2(\varphi)}{\delta^2n^2}\left\|\psi\right\|_\infty=O(n^{-2}). \end{aligned}$$

Finally we obtain

$$\lim_{n\to\infty}\left(WS_nf\right)(\rho_0)=f(\rho_0).$$

Corollary 1. Owing to the previous result one easily has

$$\lim_{n\to\infty}\left\|\left(WS_nf\right)-f\right\|_\infty=0.$$

Theorem 3. Let $f\in C_B(\mathbb{R})$. Then

$$\left\|WS_nf\right\|_\infty\leq K\left\|f\right\|_\infty,$$

is valid, where $K=M_0\left(\varphi\right)\lambda\left\|\psi\right\|_\infty$.

Proof. By (2.1) one has

$$|(WS_n f)(t)| \leq \sum_{k\in\mathbb{Z}} |\varphi(nt-k)| \int_0^\lambda \left| f\left(\frac{\tau+k}{n}\right)\right| |\psi(\tau)|\, d\tau$$

and hence follows

$$\begin{aligned} |(WS_n f)(t)| &\leq \sum_{k\in\mathbb{Z}} |\varphi(nt-k)| \, \|f\|_\infty \, \|\psi\|_\infty \, \lambda \\ &\leq M_0(\varphi) \, \|f\|_\infty \, \|\psi\|_\infty \, \lambda. \end{aligned}$$

Now, for $C_B[0,\infty)$ we denote $W^2 = \{\xi \in C_B[0,\infty) : \xi', \xi'' \in C_B[0,\infty)\}$. Consider the Peetre's $K-$functional as;

$$K_2(\sigma,\delta) := \inf_{\xi\in W^2}\{\|\sigma-\xi\|_\infty + \delta \, \|\xi''\|_\infty\}, \delta > 0, \tag{3.2}$$

satisfies

$$Y^{-1}\omega_2(\sigma,\sqrt{\delta}) \leq K_2(\sigma,\delta) \leq Y\omega_2(\sigma,\sqrt{\delta}), \tag{3.3}$$

where $Y > 0$ and

$$\omega_2(\sigma,\sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,\infty)} |\, \sigma(x+2h) - 2\sigma(x+h) + \sigma(x) \,| \tag{3.4}$$

is the modulus of smoothness of f.

Theorem 4. Let $f \in C_B(\mathbb{R})$. Then

$$\lim_{n\to\infty} (WS_n f)(x) = f(x),$$

and

$$|(WS_n f)(x) - f(x)| \leq (K+1)\, K_2\left(f; \frac{M_2(\varphi) + \lambda^2 M_0(\varphi) + 2\lambda M_1(\varphi)}{n^2}\right),$$

are valid, where $K = \lambda \, \|\psi\|_\infty$.

Proof. Let $\xi \in W^2$. We have

$$\xi(t) = \xi(x) + \xi'(x)(t-x) + \int_x^t (t-v)\xi''(v)dv, \;\; t \in [0,\infty).$$

In view of Remark 2 and (3.1), applying WS_n, we have

$$\begin{aligned}
\left|\left(WS_n\xi\right)(x)-\xi(x)\right| &= \left|\left(WS_n\left(\xi'(x)(t-x)+\int_x^t(t-v)\xi''(v)dv\right)\right)(x)\right| \\
&\le \sum_{k\in\mathbb{Z}}\left|\varphi(nx-k)\right|\int_0^\lambda\left[\int_x^{\frac{\tau+k}{n}}\left|\frac{\tau+k}{n}-v\right|\left|\xi''(v)\right|dv\right]\left|\psi(\tau)\right|d\tau \\
&\le \lambda\left\|\psi\right\|_\infty\left\|\xi''\right\|_\infty\sum_{k\in\mathbb{Z}}\left|\varphi(nx-k)\right|\left(\frac{\lambda+k}{n}-x\right)^2 \\
&= \lambda\left\|\psi\right\|_\infty\left\|\xi''\right\|_\infty\sum_{k\in\mathbb{Z}}\left|\varphi(nx-k)\right|\left[\left(\frac{k}{n}-x\right)^2+\frac{\lambda^2}{n^2}+2\frac{\lambda}{n}\left(\frac{k}{n}-x\right)\right] \\
&\le \frac{\lambda\left\|\psi\right\|_\infty\left\|\xi''\right\|_\infty}{n^2}\left[M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)\right].
\end{aligned}$$

Using (3.2), the last inequality yields

$$\left|\left(WS_nf\right)(x)-f(x)\right| \le \inf_{\xi\in W^2}\left\{\left\|WS_n(f-\xi)\right\|_\infty+\left\|f-\xi\right\|_\infty+\left|\left(WS_n\xi\right)(x)-\xi(x)\right|\right\}$$

$$\begin{aligned}
&\le \inf_{\xi\in W^2}\left\{\left(\lambda\left\|\psi\right\|_\infty+1\right)\left\|f-\xi\right\|_\infty+\frac{\lambda\left\|\psi\right\|_\infty\left[M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)\right]}{n^2}\left\|\xi''\right\|_\infty\right\} \\
&\le (K+1)\inf_{\xi\in W^2}\left\{\left\|f-\xi\right\|_\infty+\frac{M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)}{n^2}\left\|\xi''\right\|_\infty\right\} \\
&= (K+1)K_2\left(f;\frac{M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)}{n^2}\right).
\end{aligned}$$

Theorem 5. Let $f\in C_B(\mathbb{R})$ and $\mu\in(0,2)$ be fixed. Then

$$\omega_2(f;t)=\mathcal{O}(t^\mu)\Rightarrow\left|\left(WS_nf\right)(x)-f(x)\right|=\mathcal{O}(1/n)^\mu$$

holds true.

Proof. Owing to (3.3) and Theorem 4 one gets

$$\begin{aligned}
\left|\left(WS_nf\right)(x)-f(x)\right| &\le (K+1)K_2\left(f;\frac{M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)}{n^2}\right) \\
&\le (K+1)C\omega_2\left(f;\sqrt{\frac{M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)}{n^2}}\right) \\
&\le (K+1)C\left(\frac{M_2(\varphi)+\lambda^2M_0(\varphi)+2\lambda M_1(\varphi)}{n^2}\right)^{\mu/2}.
\end{aligned}$$

Theorem 6. Let $f\in L^1(\mathbb{R})$, then

$$\left\|WS_nf\right\|_1\le K\left\|f\right\|_1,$$

holds true, where $K=nh\left\|\psi\right\|_\infty\left\|\varphi\right\|_1$, $h:=\lfloor\lambda\rfloor+1$ and $\lfloor\bullet\rfloor$ is the floor function.

Proof.

$$\begin{aligned}\int\limits_{\mathbb{R}} |(WS_nf)(t)|\,dt &= \int\limits_{\mathbb{R}} \left|\sum_{k\in\mathbb{Z}} \varphi(nt-k)\int\limits_0^{\lambda} f\left(\frac{\tau+k}{n}\right)\psi(\tau)d\tau\right| dt \\ &\leq \int\limits_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi(nt-k)| \int\limits_0^{\lambda} \left|f\left(\frac{\tau+k}{n}\right)\right| |\psi(\tau)|\,d\tau dt \\ &\leq n\,\|\psi\|_\infty \sum_{k\in\mathbb{Z}} \int\limits_{\frac{k}{n}}^{\frac{\lambda+k}{n}} |f(u)|\,du \left(\int\limits_{\mathbb{R}} |\varphi(n\tau-k)|\,d\tau\right) \\ &\leq n\,\|\psi\|_\infty \|\varphi\|_1 \sum_{k\in\mathbb{Z}} \int\limits_{\frac{k}{n}}^{\frac{\lambda+k}{n}} |f(u)|\,du.\end{aligned}$$

Setting $h := \lfloor \lambda \rfloor + 1$. Hence we have

$$\begin{aligned}\int\limits_{\mathbb{R}} |(WS_nf)(t)|\,dt &\leq n\,\|\psi\|_\infty \|\varphi\|_1 \sum_{k\in\mathbb{Z}} \int\limits_{\frac{k}{n}}^{\frac{h+k}{n}} |f(u)|\,du \\ &\leq nh\,\|\psi\|_\infty \|\varphi\|_1 \|f\|_1 \\ &= : K\,\|f\|_1\,,\end{aligned}$$

here $K = nh\,\|\psi\|_\infty \|\varphi\|_1$.

Theorem 7. Let $f \in L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$. Then

$$\|WS_nf\|_p \leq K_p\,\|f\|_p\,,$$

holds, where $K_p = n\,\|\psi\|_\infty \|\varphi\|_1^{1/p} h^{1/p} M_0^{(p-1)/p}(\varphi) > 0$.

Proof. Clearly

$$\begin{aligned}\left(\int\limits_{\mathbb{R}} |(WS_nf)(t)|^p\,dt\right)^{1/p} &= \left(\int\limits_{\mathbb{R}} \left|n\sum_{k\in\mathbb{Z}} \varphi(nt-k)\int\limits_{\mathbb{R}} f(\tau)\,\psi(n\tau-k)d\tau\right|^p dt\right)^{1/p} \\ &\leq n\left(\int\limits_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} |\varphi(nt-k)| \int\limits_{\mathbb{R}} |f(\tau)\,\psi(n\tau-k)|\,d\tau\right)^p dt\right)^{1/p} \\ &= : W.\end{aligned}$$

Using Jensen inequality one gets

$$\begin{aligned} W^p &\le nM_0^{p-1}(\varphi) \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\varphi(nt-k)| \left(\int_{\mathbb{R}} |f(\tau)\,\psi(n\tau-k)|\, d\tau \right)^p dt \\ &= nM_0^{p-1}(\varphi) \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} |f(\tau)|^p \, |\psi(n\tau-k)|^p \int_{\mathbb{R}} |\varphi(nt-k)|\, dt \right) d\tau \\ &\le nM_0^{p-1}(\varphi) \, \|\psi\|_\infty \, \|\varphi\|_1 \sum_{k\in\mathbb{Z}} \int_{\frac{k}{n}}^{\frac{\lambda+k}{n}} |f(\tau)|^p \, d\tau. \end{aligned}$$

So one has

$$W^p \le n^p \, \|\psi\|_\infty^p \, \|\varphi\|_1 \, h \, \|f\|_p^p \, M_0^{p-1}(\varphi),$$

and

$$\begin{aligned} \|WS_n f\|_p &\le \left(n^p \, \|\psi\|_\infty^p \, \|\varphi\|_1 \, h \, \|f\|_p^p \, M_0^{p-1}(\varphi) \right)^{1/p} \\ &= K_p \, \|f\|_p, \end{aligned}$$

here $K_p = n \, \|\psi\|_\infty \, \|\varphi\|_1^{1/p} \, h^{1/p} M_0^{(p-1)/p}(\varphi)$.

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