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SOME FRACTIONAL ORDER DIFFERENTIAL EQUATIONS BY USING CAPUTO FABRIZIO DERIVATIVE

RAVINDER KUMAR, ASHOK KUMAR BADSARA, AND VIRENDRA SINGH CHOUHAN*

Abstract. In this paper, we study for the fractional order linear partial differential equation with the help of the Caputo–Fabrizio (CF) fractional derivative. Moreover, the approximate solution of the linear partial differential is obtained by using the Homotopy perturbation transform method. The fractional differential equations has approximate solutions, which were found. These solutions converged to exact results at a reduced computational cost rapidly. Additionally, the approach taken in this work is more broadly applicable, which enables our findings to be more comprehensive and address a number of novel and well-known fractional problems in the literature.

1. Introduction

Fractional calculus is a branch of mathematical analysis that extends the concept of derivatives and integrals to non-integer orders. In classical calculus, we deal with integer-order derivatives and integrals, such as first derivatives, second derivatives, and so on. These procedures are generalized to any real or complex order, not only whole numbers, by fractional calculus. linear fractional partial differential equations (LFPDEs) are useful because they can theoretically explain a wide range of phenomena in mathematical physics and many other scientific domains in the modern world. Biology, sociology, medicine, hydrodynamics, computer modeling, chemical kinetics, aerodynamics, rheology, diffusion, electrostatics, electrodynamics, control theory, fluid mechanics, analytical chemistry, and other fields have all benefited from the successful application of LFPDEs ([1], [2], [3], [4], [5], [6], [7], [8], [9]). Numerous scholars have put forth novel combination and hybrid approaches in recent years to investigate various LFPDE types. For example,the Homotopy perturbation technique[10-11], Integral transform method [12], variational iteration method[13],Laplace transform method [14- 16], the decomposition method [17] and Modified homotopy perturbation method [18]. General-purpose problems are solved with numerical techniques.The discretization of fractional-order derivatives cannot be done using the traditional method when dealing with fraction-order derivatives. The global impact of fractional-order derivatives makes the cause clear. Ordinary derivatives are typically seen inside of integrals that present fractional-order derivatives. We first estimate the integral using a numerical method, which provides us with a linear combination of ordinary derivatives.

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Ordinary derivatives can be approximated using a traditional method once we have an equation including them. In the current study, we employ the Homotopy perturbation transform approach to provide a first-order approximation of ordinary derivatives. Future studies should consider using a higher-order approximation for ordinary derivatives. The structure of the paper is as follows: Section 1 has added the introduction based on the essential investigations on fractional derivative and numerical techniques. has included an introduction based on the fundamental research on numerical methods and fractional derivative. Section 2 contains the mathematical basic tools. section 3 A succinct explanation of the numerical method, which is the Homotopy perturbation transform approach. Section 4 we solve a different linear fractional problems. section 5 In Section 5, the findings and analysis of the approximate solution are emphasized. This study's conclusions are added in the second last section, and the essential references related to this work are cited in the previous section of this paper.

2. Preliminaries

Here are some key concepts:
Definition 2.1 For $\alpha > 0$ left (R-L) order fractional integral of α is defined as [1]

$$ {}_a\, I_t^{\alpha}\; u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\xi)^{\alpha-1} u(\xi) d\xi, \tag{2.1}$$

Definition 2.2 For $0 < \alpha < 1$ left (R-L) order fractional integral of α is given as [1]

$$ ({}_a\, D_t^{\alpha}\; u)(t) = \frac{d}{dt}({}_a\, I_t^{\alpha}\; u)(t) = \frac{\frac{d}{dt}}{\Gamma(1-\alpha)} \int_a^t (t-\xi)^{-\alpha} u(\xi) d\xi, \tag{2.2}$$

Definition 2.3 For Caputo derivative is define for $\alpha \geq 0$ & $n \in N \cup 0$ is defined as [1]:

$$ {}_0^{CF}\, D_t^{\alpha}\; u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi) \frac{d^n}{dt^n} u(\xi) d\xi, \tag{2.3}$$

where ${}_0^{CF}\, D_t^{\alpha}$ is a Caputo Fabrizio derivative.

Definition 2.4 Assume u be a function $u \in H^1(a_1, b_1),\ b_1 > 0,\ 0 < \alpha < 1$. Then, CFD is define as follows [1]:

$$ {}_0^{CF}\, D_t^{\alpha}\; u(t) = \frac{\lambda(\alpha)}{1-\alpha)} \int_0^t exp[-\frac{\alpha(1-\xi)}{1-\alpha}] u'(\xi) d\xi, t \geq 0,\ 0 < \alpha < 1, \tag{2.4}$$

with a normalize functions $\lambda(\alpha)$ which is depend on $\alpha \in \lambda(0) = \lambda(1) = 1$.
Definition 2.5 For CFD of order $0 < \alpha < 1$. is given as [1]

$$ {}_0^{CF}\, D_t^{\alpha}\; u(t) = \frac{2(1-\alpha)}{\lambda(\alpha)(2-\alpha)} u(t) + \frac{2\alpha}{\lambda(\alpha)(2-\alpha)} \int_0^t u(\xi) d\xi,\ t \geq 0, \tag{2.5}$$

where ${}_0^{CF}\, D_t^{\alpha}\; u(t) = 0$, if u is a constant function.
Definition 2.6 The Laplace transform (LT) for the (CFD) of order $0 < \alpha < 1$.

and $m \in N$ is gives as [1]

$$\begin{aligned} L\left[{}_0^{CF}\ D_t^{(m+\alpha)}\ u(t)\right](s) =& \frac{1}{1-\alpha} L[u^{m+1}(t)] L\left[exp\left(\frac{-\alpha}{(1-\alpha)}t\right)\right] \\ =& \frac{s^{m+1}L[u(t)] - s^m u(0) - s^{m-1}u'(0)\ldots - u^m(0)}{s+\alpha(1-s)} \end{aligned} \tag{2.6}$$

In particular, we have

$$L\left[{}_0^{CF}\ D_t^{(m+\alpha)}\ u(t)\right](s) = \frac{sL(u(t))}{s+\alpha(1-s)},\ m=0,$$

$$L\left[{}_0^{CF}\ D_t^{(m+\alpha)}\ u(t)\right](s) = \frac{s^2L(u(t)) - su(o) - u'(0)}{s+\alpha(1-s)},\ m=1.$$

3. General description of FHPTM using Caputo-Fabrizio type operator:

Let's as consider the following (NPDEs) along with Caputo-Fabrizio operator:

$${}_0^{CF}\ D_t^{m+\alpha} u(x,t) + \beta u(x,t) + \varphi u(x,t) = k(x,t),\ n-1 < \alpha + m \leq n, \tag{3.1}$$

such that initial conditions(I.C)

$$\frac{\partial^l u(x,0)}{\partial t^l} = f_l(x),\ l = 0,1,2,\ldots n-1. \tag{3.2}$$

Now,be applying the (LT) on both Eq. (3.1), we get

$$L[u(x,t)] = \Theta(x,s) - \left(\frac{s+\alpha(1-s)}{s^{n+1}}\right) L[\beta u(x,t) + \varphi u(x,t)]. \tag{3.3}$$

where

$$\Theta(x,s) = \frac{1}{s^{m+1}}[s^m f_0(x) + s^{m-1}f_1(x) + \ldots + f_m(x)] + \frac{s+\alpha(1-s)}{s^{n+1}}\tilde{k}(x,s). \tag{3.4}$$

Apply Inverse Laplace transformation on both side Eq.(3.3),we have

$$u(x,t) = \Theta(x,s) - L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{n+1}}\right) L[\beta u(x,t) + \varphi u(x,t)]\right], \tag{3.5}$$

$\Theta(x,s)$ is a source term,the initial condition of solution $u(x,t)$ may be extended into an infinite sequence used regular homotopy perturbation method as follows:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \tag{3.6}$$

where $u_m(x,t)$ are known functions as follows:

$$\varphi u(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t). \tag{3.7}$$

The poly. $H_n(x,t)$ are define as [13]

$$H_m(u_0,u_1,u_2,\dots u_n) = \frac{1}{n!}\frac{\partial^m}{\partial z^m}\left[\left(\sum_{m=0}^{\infty} p^i u_i\right)\right]_{p=0},\ m=0,1,2,\dots; \tag{3.8}$$

substitute Eq.(3.6) and Eq.(3.7) into Eq.(3.5) we are getting

$$\sum_{m=0}^{\infty} u_m(x,t) = \Theta(x,s) - pL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)L\left[\beta\sum_{m=0}^{\infty}p^m u_m(x,t)+\sum_{n=0}^{\infty}p^m H_m\right]\right] \tag{3.9}$$

Comparing the coefficients of $p^0,\ p^1,\ p^2,\ p^3$ and p^4 as follows:

$$\begin{aligned}
p^0 : u_0(x,t) &= \Theta(x,s),\\
p^1 : u_1(x,t) &= -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)L[\beta u_0(x,t)+H_0(u)]\right]\\
p^2 : u_2(x,t) &= -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)L[\beta u_1(x,t)+H_1(u)]\right]\\
p^3 : u_3(x,t) &= -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)L[\beta u_2(x,t)+H_2(u)]\right]\\
&\vdots\\
p^{m+1} : u_{m+1}(x,t) &= -L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^{m+1}}\right)L[\beta u_{m+1}(x,t)+H_{m+1}(u)]\right].
\end{aligned}$$

The solution is written as follows:

$$u(x,t) = u_0 + u_1 + u_2 + ..., \tag{3.10}$$

4. Numerical experiments

We'll look at some numerical examples for the linear fractional partial differential equations

Example 4.1 One-dimensional wave equation as follows: [17]

$${}_0^{CF}D_{tt}^{\alpha}\, u = u_{xx},\ 0 < x < \pi,\ 1 < \alpha \le 2, \tag{4.1}$$

with boundary condition

$$u(0,t) = 0,\ u(\pi,t) = 0, \tag{4.2}$$

Initial condition

$$u(x,0) = \sin x,\ u_t(x,0) = 0. \tag{4.3}$$

Taking (LT) Laplace on Eq.(4.1) and using conditions given in Eqs. (4.3), we obtain

$$L[u(x,t)] = \frac{1}{s}\sin x + \left(\frac{s+\alpha(1-s)}{s^2}\right)L\left[u_{xx}\right]. \tag{4.4}$$

Inverse Laplace Transform for Eq.(4.4)

$$u(x,t) = \sin x + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)L\left[u_{xx}\right]\right]. \tag{4.5}$$

Now, we apply the FHPTM, we get

$$\sum_{m=0}^{\infty} u_m(x,t) = \sin x + pL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{xx}\right]\right], \quad (4.6)$$

Firmly facing the above conditions, we get

$$p^0 : u_0(x,t) = \sin x,$$

$$p^1 : u_1(x,t) = -\left(t(1-\alpha) + \frac{t^2\alpha}{2}\right)\sin x,$$

$$p^2 : u_2(x,t) = \left(\frac{t^4\alpha^2}{24} + \frac{1}{2}t^2\left(1-2\alpha+\alpha^2\right) - \frac{1}{3}t^3\left(-\alpha+\alpha^2\right)\right)\sin x,$$

$$p^3 : u_3(x,t) = \left(-\frac{1}{720}t^6\alpha^3 + \frac{1}{6}t^3\left(-1+3\alpha-3\alpha^2+\alpha^3\right) - \frac{1}{8}t^4\left(\alpha-2\alpha^2+\alpha^3\right) + \frac{1}{40}t^5\left(-\alpha^2+\alpha^3\right)\sin x,\right.$$

$$p^4 : u_4(x,t) = \left(\frac{t^8\alpha^4}{40320} + \frac{1}{24}t^4\left(1-4\alpha+6\alpha^2-4\alpha^3+\alpha^4\right) - \frac{1}{30}t^5\left(-\alpha+3\alpha^2-3\alpha^3+\alpha^4\right) + ...,.\sin x,\right.$$

$$p^5 : u_5(x,t) = \left(-\frac{t^{10}\alpha^5}{3628800} + \frac{1}{120}t^5\left(-1+5\alpha-10\alpha^2+10\alpha^3-5\alpha^4+\alpha^5\right) + \frac{1}{504}t^7\left(-\alpha^2+3\alpha^3-3\alpha^4+\alpha^5\right) - ..., \sin x.\right.$$

$$\vdots$$

so the solution $u(x,t)$ is written as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + ..., \quad (4.7)$$

Example 4.2 consider two-dimensional diffusion equation as [17]

$${}_{0}^{CF}D_t^{\alpha}\, u = u_{xx} + u_{yy},\ 0 < x < \pi,\ 0 < y < \pi,\ 0 < \alpha \leq 1, \quad (4.8)$$

with boundary condition

$$\begin{cases} u(0,y,t) = 0, \\ u(\pi,y,t) = 0, \\ u(x,0,t) = 0, \\ u(x,\pi,t) = 0, \end{cases} \quad (4.9)$$

and initial condition

$$u(x,0) = \sin x\ \sin y. \quad (4.10)$$

Now by applying Laplace Transform on Eq.(4.8) and using conditions given in Eqs. (4.10), we obtain

$$L[u(x,t)] = \frac{1}{s}\sin x \sin y + \left(\frac{s+\alpha(1-s)}{s}\right) L\left[u_{xx} + +u_{yy}\right]. \quad (4.11)$$

Inverse Laplace Transform for Eq.(4.11)

$$u(x,t) = \sin x \sin y + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[u_{xx} + u_{yy}(x,y)\right]\right]. \tag{4.12}$$

Now, we apply the FHPTM, we have

$$\sum_{m=0}^{\infty} u_m(x,t) = \sin x \sin y +$$
$$pL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{xx} + \left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{yy}\right]\right] \tag{4.13}$$

Firmly facing the above conditions, as follows

$$p^0 : u_0(x,t) = \sin x \sin y,$$
$$p^1 : u_1(x,t) = -2(1-\alpha+t\alpha)\sin x \sin y,$$
$$p^2 : u_2(x,t) = 4\left(1-2\alpha+\alpha^2+\frac{t^2*\alpha^2}{2} - 2t\left(-\alpha+\alpha^2\right)\right)\sin x \sin y,$$
$$p^3 : u_3(x,t) = 8\left(1-3\alpha+3\alpha^2-\alpha^3+\frac{t^3*\alpha^3}{6} + 3t\left(\alpha-2\alpha^2+\alpha^3\right) - \frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\right)\sin x \sin y,$$
$$p^4 : u_4(x,t) = 16\left(1-4\alpha+6\alpha^2-4\alpha^3+\alpha^4+\frac{t^4*\alpha^4}{24} - 4t\left(-\alpha+3\alpha^2-3\alpha^3+\alpha^4\right)+...,\right)\sin x \sin y$$
$$p^5 : u_5(x,t) = -32\Bigg(1-5\alpha+10\alpha^2-10\alpha^3+5\alpha^4-\alpha^5+\frac{t^5*\alpha^5}{120} + 5t\left(\alpha-4\alpha^2+6\alpha^3-4\alpha^4\right)-...,$$
$$\sin x \sin y.$$
$$\vdots$$

so the solution $u(x,t)$ is written as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + ..., \tag{4.14}$$

Example 4.3 consider 3D space diffusion equation as follows [17]

$${}_0^{CF}D_t^{\alpha}\, u = u_{xx} + u_{yy} + u_{zz},,\ 0 < x, y, z < \pi,\ 0 < \alpha \leq 1, \tag{4.15}$$

with boundary condition

$$\begin{cases} u(0,y,z,t) = u(\pi,y,z,t), \\ u(x,0,z,t) = u(x,\pi,z,t), \\ u(x,y,0,t) = u(x,y,\pi,t), \end{cases} \tag{4.16}$$

and Initial condition

$$u(x,y,z,0) = \sin x \sin y \sin z. \tag{4.17}$$

Now by applying Laplace Transform on Eq.(4.15) and using conditions given in Eqs. (4.17), we have

$$L[u(x,t)] = \frac{1}{s}\sin x \sin y \sin z + \left(\frac{s+\alpha(1-s)}{s}\right) L\left[u_{xx} + +u_{yy} + u_{zz}\right]. \quad (4.18)$$

Inverse Laplace Transform for Eq.(4.18)

$$u(x,t) = \sin x \sin y \sin z + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right) L\left[u_{xx} + u_{yy} + u_{zz}\right]\right]. \quad (4.19)$$

Now, we apply the FHPTM

$$\begin{aligned}\sum_{m=0}^{\infty} u_m(x,t) &= \sin x \sin y \sin z \\ &+ pL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s}\right)\right. \\ &L\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{xx} + \left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{yy} + \left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{zz}\right],\end{aligned} \quad (4.20)$$

Firmly facing the above conditions, we get

$$\begin{aligned}
p^0 : u_0(x,t) =& \sin x \sin y \sin z,\\
p^1 : u_1(x,t) =& -3(1-\alpha+t\alpha)\sin x \sin y \sin z,\\
p^2 : u_2(x,t) =& 9\left(1-2\alpha+\alpha^2+\frac{t^2\alpha^2}{2}-2t\left(-\alpha+\alpha^2\right)\right)\sin x \cos y \sin z,\\
p^3 : u_3(x,t) =& -27\left(1-3\alpha+3\alpha^2-\alpha^3+\frac{t^3\alpha^3}{6}+3t\left(\alpha-2\alpha^2+\alpha^3\right)-\right.\\
& \frac{3}{2}t^2\left(-\alpha^2+\alpha^3\right)\sin x \cos y \sin z,\\
p^4 : u_4(x,t) =& 81\left(1-4\alpha+6\alpha^2-4\alpha^3+\alpha^4+\frac{t^4\alpha^4}{24}-\right.\\
& 4t\left(-\alpha+3\alpha^2-3\alpha^3+\alpha^4\right)+..., \sin x \cos y \sin z,\\
p^5 : u_5(x,t) =& -243\left(1-5\alpha+10\alpha^2-10\alpha^3+5\alpha^4-\alpha^5+\frac{t^5\alpha^5}{120}+\right.\\
& 5t\left(\alpha-4\alpha^2+6\alpha^3-4\alpha^4+\alpha^5\right)-..., \sin x \cos y \sin z.\\
& \vdots
\end{aligned}$$

so the solution $u(x,t)$ is written as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + ..., \quad (4.21)$$

Example 4.4 considering the wave equation as follows [17]

$${}_0^{CF}D_{tt}^{\alpha}\, u = u_{xx},\ 1 < \alpha \leq 2, \quad (4.22)$$

B.C

$$u(0,t) = \sin t,\ u(\pi,t) = \pi, \quad (4.23)$$

I.C

$$u(x,0) = x,\ u_t(x,0) = \cos x. \tag{4.24}$$

Now by applying Laplace Transform on Eq.(4.22) and using conditions given in Eqs. (4.24), we get:

$$L[u(x,t)] = \frac{1}{s}x + \frac{1}{s^2}\cos x + \left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[u_{xx}\right]. \tag{4.25}$$

Inverse Laplace Transform for Eq.(4.25)

$$u(x,t) = x + t\cos x + L^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[u_{xx}\right]\right]. \tag{4.26}$$

Now, applying the FHPTM, we have

$$\sum_{m=0}^{\infty} u_m(x,t) = x + t\cos x + pL^{-1}\left[\left(\frac{s+\alpha(1-s)}{s^2}\right) L\left[\left(\sum_{m=0}^{\infty} p^m u_m(x,t)\right)_{xx}\right]\right], \tag{4.27}$$

Firmly facing the above conditions, we define

$$\begin{aligned}
p^0 : u_0(x,t) =& x + t\cos x,\\
p^1 : u_1(x,t) =& -\left(\frac{1}{2}t^2(1-\alpha) + \frac{t^3\alpha}{6}\right)\cos x,\\
p^2 : u_2(x,t) =& \left(\frac{t^5\alpha^2}{120} + \frac{1}{6}t^3\left(1-2\alpha+\alpha^2\right) - \frac{1}{12}t^4\left(-\alpha+\alpha^2\right)\right)\cos x,\\
p^3 : u_3(x,t) =& \Big(-\frac{t^7\alpha^3}{5040} + \frac{1}{24}t^4\left(-1+3\alpha-3\alpha^2+\alpha^3\right) -\\
& \frac{1}{40}t^5\left(\alpha-2\alpha^2+\alpha^3\right) + \frac{1}{240}t^6\left(-\alpha^2+\alpha^3\right)\cos x,\\
p^4 : u_4(x,t) =& \Big(\frac{t^9\alpha^4}{362880} + \frac{1}{120}t^5\left(1-4\alpha+6\alpha^2-4\alpha^3+\alpha^4\right) -\\
& \frac{1}{180}t^6\left(-\alpha+3\alpha^2-3\alpha^3+\alpha^4\right) + ..., \cos x.\\
& \vdots
\end{aligned}$$

so the solution $u(x,t)$ is written as

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + u_4 + ..., \tag{4.28}$$

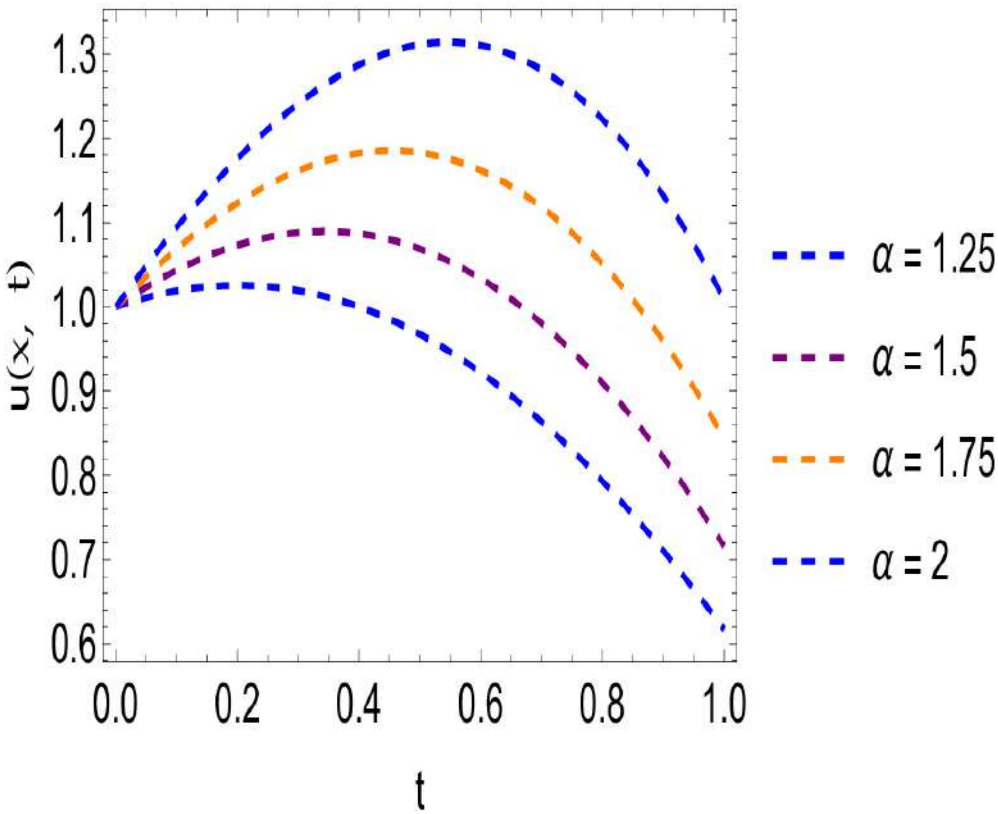

FIGURE 1. Plots of the approximate solutions u(x,t) for various values of α with $x = \frac{\pi}{2}$.

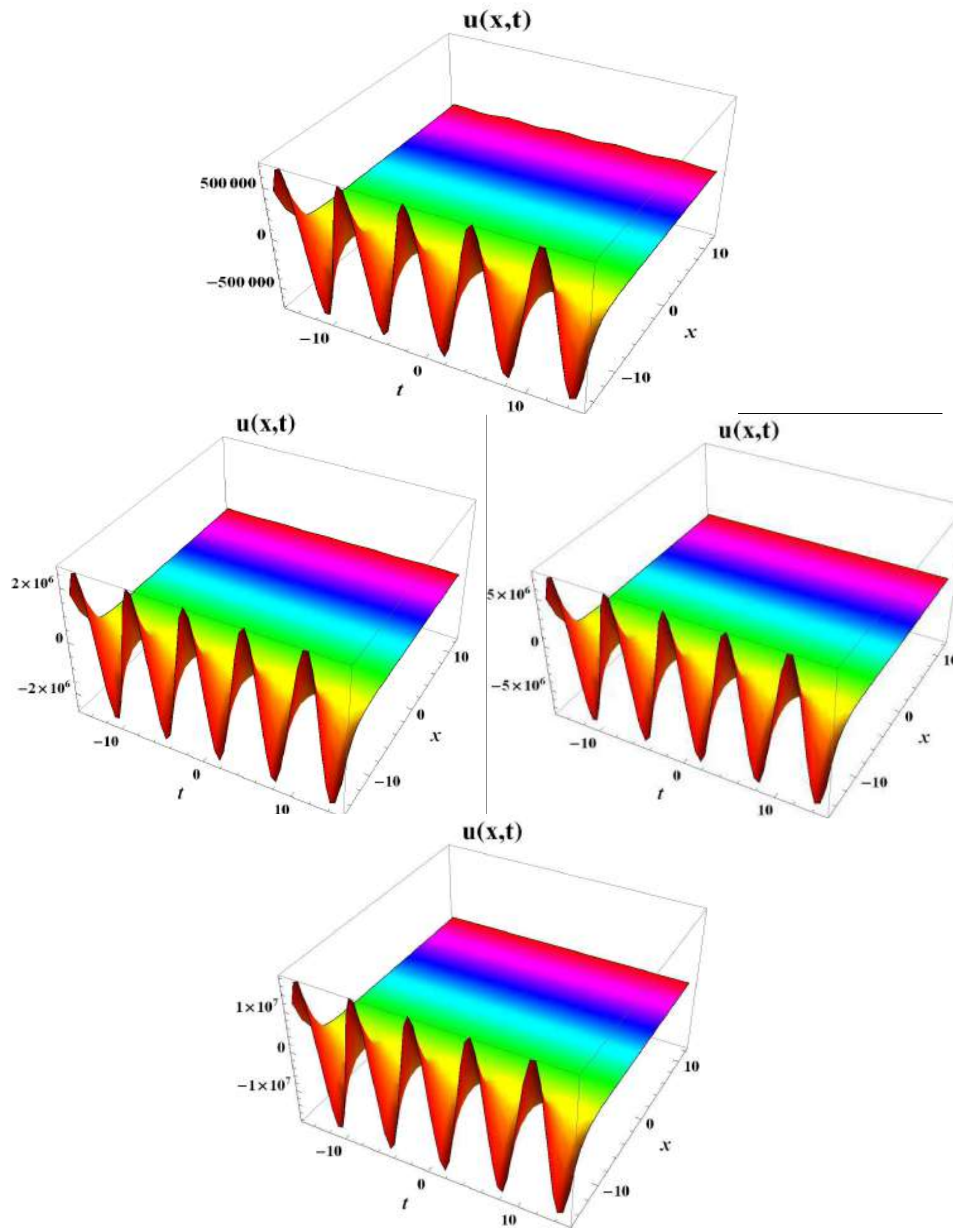

FIGURE 2. For (a) $\alpha = 1.25$, (b) $\alpha = 1.5$, (c) $\alpha = 1.75$ and (d) $\alpha = 2$, the surface graph of the approximate solutions of Eq. (4.1)

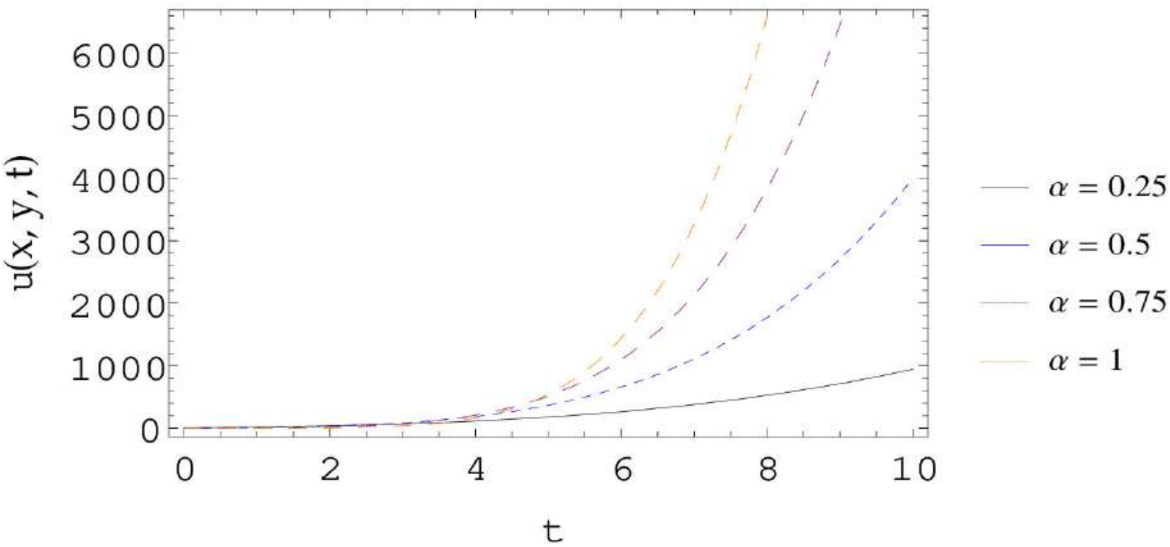

FIGURE 3. Plots of the approximate solutions u(x, y, t) for various values of α with $x = \frac{\pi}{2}$ and $y = \frac{3\pi}{2}$.

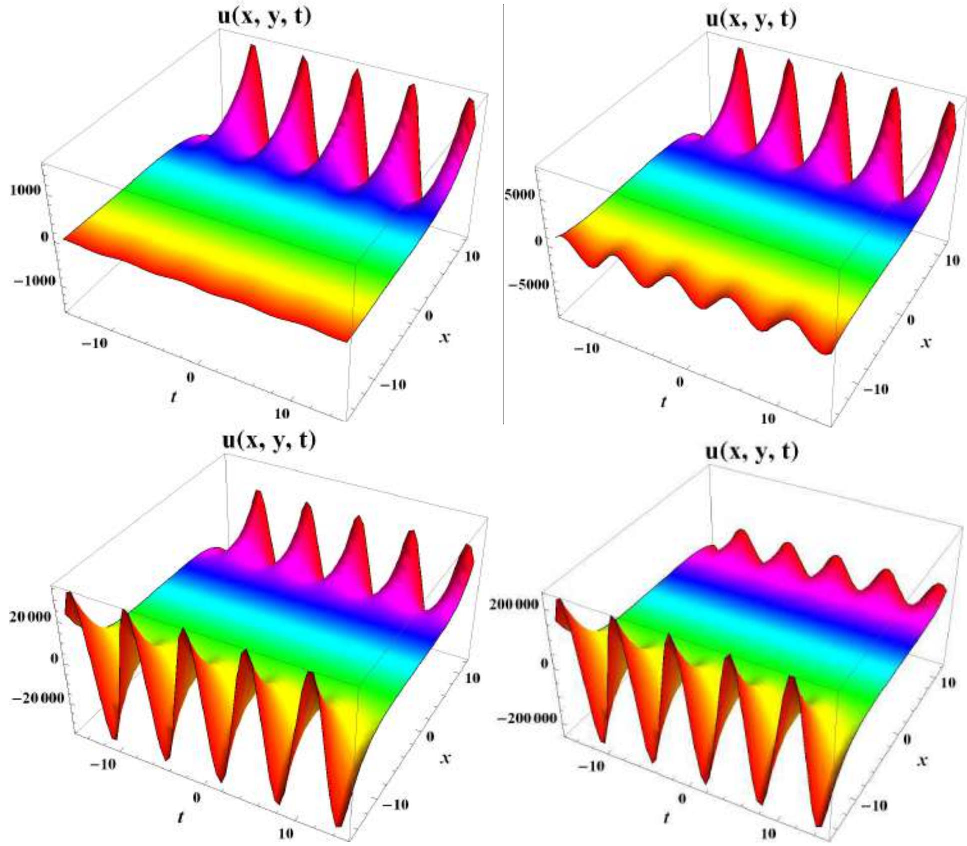

FIGURE 4. For (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$, the surface graph of the approximate solutions of Eq. (4.8)

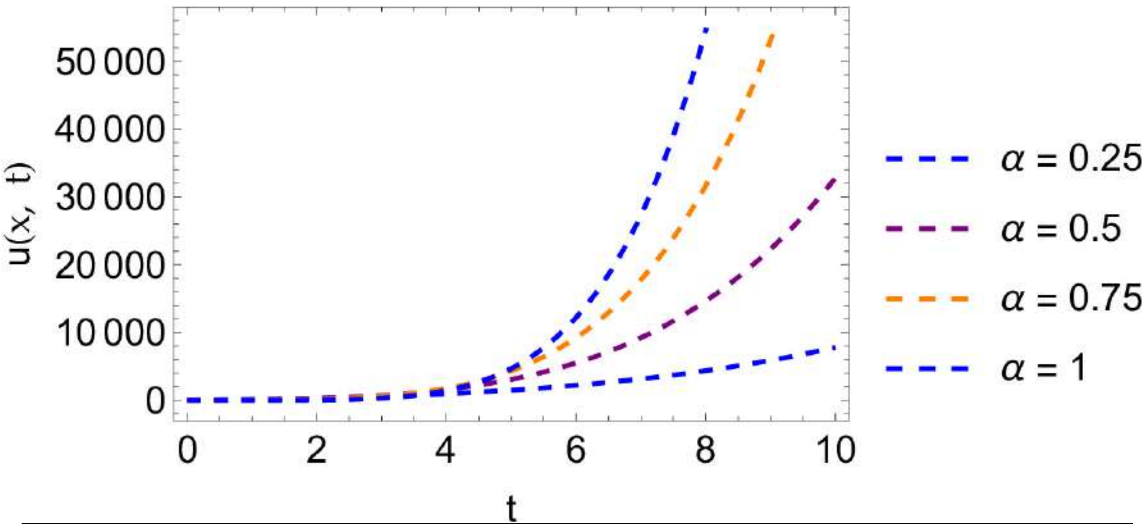

FIGURE 5. Plots of the approximate solutions u(x, y, t) for various values of α with $x = \frac{\pi}{2}$, $y = \frac{3\pi}{2}$ and $z = \frac{5\pi}{2}$.

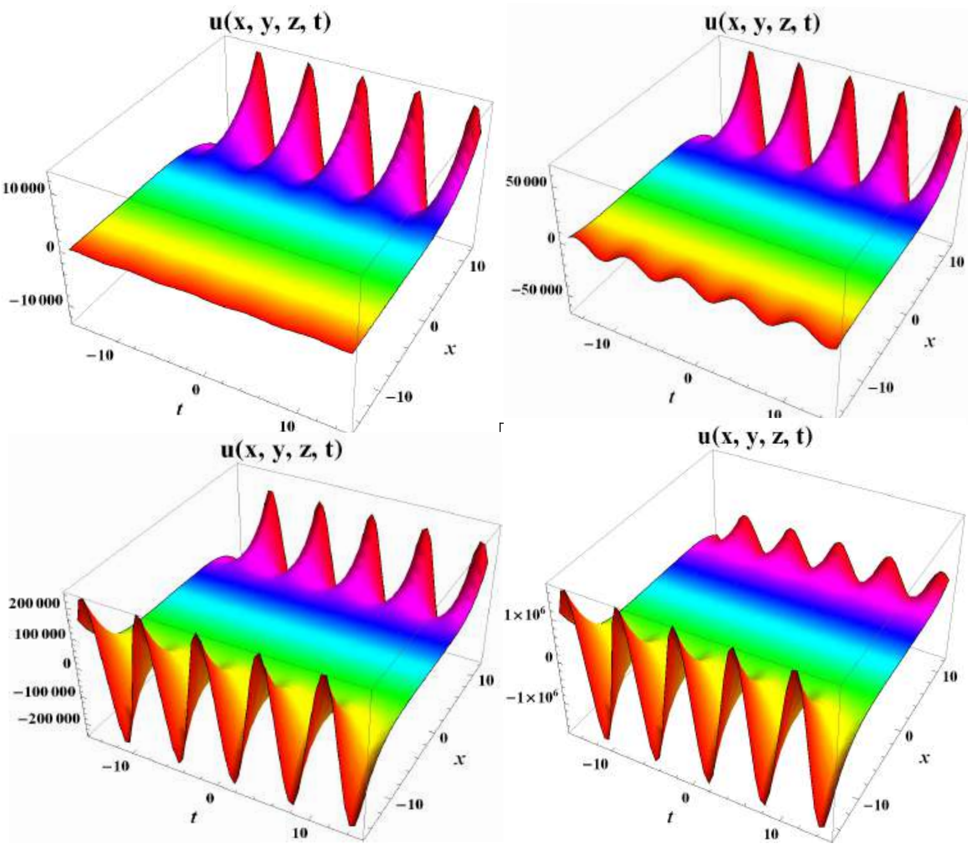

FIGURE 6. For (a) $\alpha = 0.25$, (b) $\alpha = 0.5$, (c) $\alpha = 0.75$ and (d) $\alpha = 1$, the surface graph of the approximate solutions of Eq. (4.15)

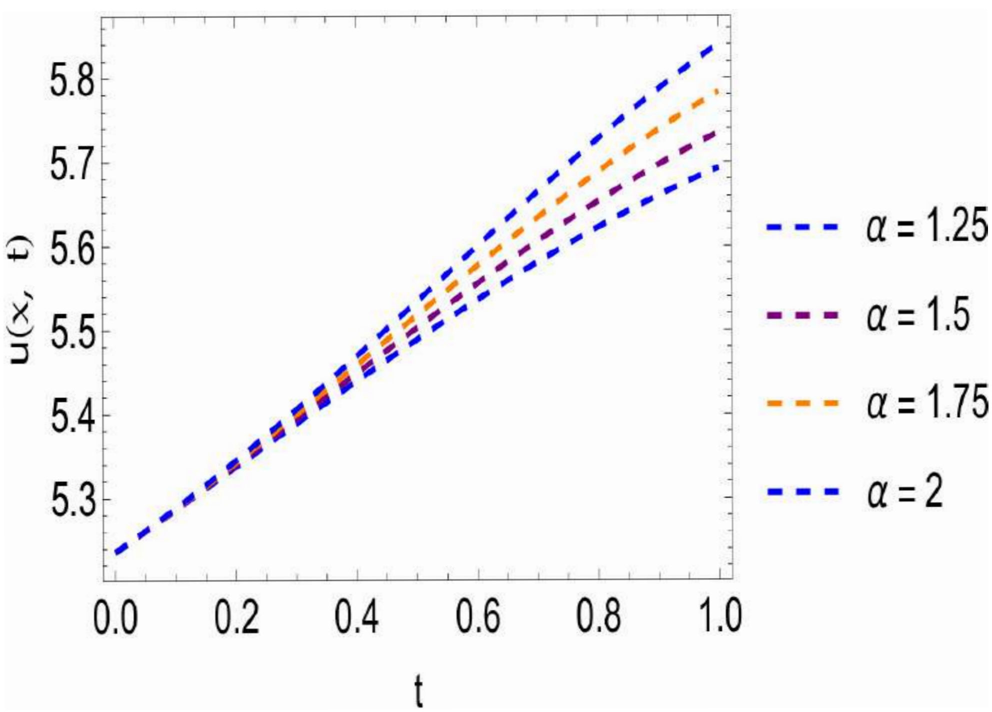

FIGURE 7. Plots of the approximate solutions u(x,t) for various values of α with $x = \frac{5\pi}{3}$.

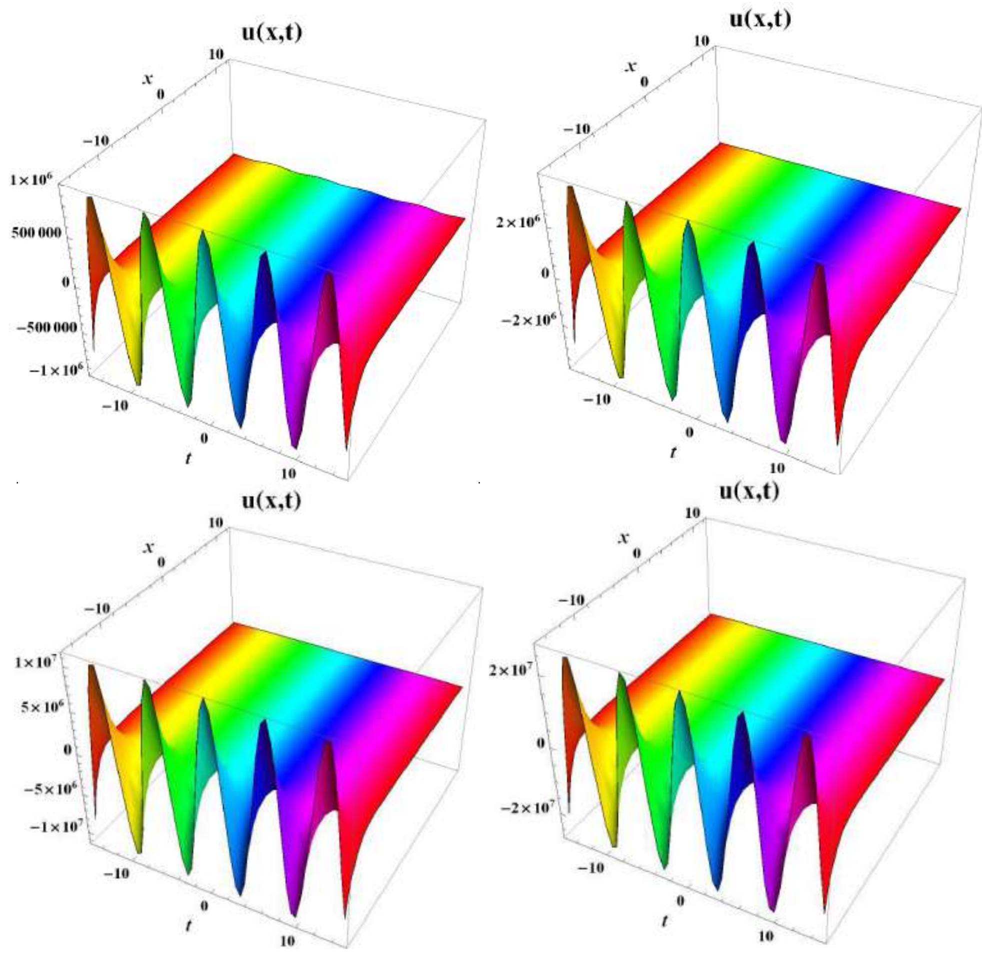

FIGURE 8. For (a) $\alpha = 1.25$, (b) $\alpha = 1.5$, (c) $\alpha = 1.75$ and (d) $\alpha = 2$, the surface graph of the approximate solutions of Eq. (4.22)

5. Results and discussion

Fig.1 shows the comparison of approximate solution for Eq. (4.1) attained by FHPTM versus t for variousvalues of α. Fig.2 (a)- (d) shows the profile of the five order approximation solution for 1D- fractional wave equation. for $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ at $\alpha = 1.25,\ 1.50,\ 1.75$, and $\alpha = 1.90$. for Eq.(4.1) by the application of initial condition represented by the Eq.(4.8) of $u(x,t)$. Fig.2 depicts the solitary wave nature of the approximate solution produced by FHPTM for the second order fractional wave equation. Fig.3 shows the comparison of approximate solution for Eq. (4.8) attained by FHPTM versus t for different values of α. Fig.4 (a)- (d) shows the profile of the five order approximation solution for two-dimensional fractional heat equation. for $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ at $\alpha = 0.25,\ 0.50,\ 0.75$, and $\alpha = 0.90$. for Eq.(4.8) by the application of initial condition represented by the Eq.(4.14) of $u(x,y,t)$. Fig.4 depicts the solitary wave nature of the approximate solution produced by FHPTM for the 2D- fractional heat equation. Fig.5 shows the comparison of approximate solution for Eq. (4.15) attained by FHPTM versus t for different values of α. Fig.6 (a)- (d) shows the profile of the five order approximation solution for 3D- fractional heat equation. for $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ at $\alpha = 0.25,\ 0.50,\ 0.75$, and $\alpha = 0.90$. for Eq.(4.15) by the application of initial condition represented by the Eq.(4.21) of $u(x,y,z,t)$. Fig.6 depicts the solitary wave nature of the approximate solution produced by FHPTM for the 3D- fractional heat equation. Fig.7 shows the comparison of approximate solution for Eq. (3.22) attained by FHPTM versus t for different values of α. Fig.8 (a)- (d) shows the profile of the five order approximation solution for 1D- fractional wave equation. for $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ at $\alpha = 1.25,\ 1.50,\ 1.75$, and $\alpha = 1.90$. for Eq.(4.22) by the application of initial condition represented by the Eq.(4.28) of $u(x,t)$. Fig.8 depicts the solitary wave nature of the approximate solution produced by FHPTM for the 1D fractional wave equation.

6. Conclusion

In this paper, we used the HPTM to obtain the approximate solutions of 1D, 2D and 3D fractional differential equations. We found that the series of solutions converged to exact solutions, which is consistent with the findings in [9-10]. For different values of α, the approximate series solutions in Figures 1, 3, 5, and 7 were essentially equivalent. However, increasing the values of m can enhance the approximate series solution. CFD aids in the rapid solution of fractional orders, allowing the series solution to approach the precise solution.Finally, the suggested method's ease of implementation suggests that it could be modified to solve additional FPDEs that arise in applied science.

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Ravinder Kumar: Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India.
Email address: ravinder.202505032@muj.manipal.edu

Ashok Kumar Badsara: Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India.
Email address: ashok.202505029@muj.manipal.edu

Virendra Singh Chouhan: Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India.
Email address: virendrasingh.chouhan@jaipur.manipal.edu(*Corresponding author)