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EMBEDDING OF SEMIRINGS IN RINGS

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Abstract. Let $(S, +, .)$ be a semiring which is additively cancellative and commutative with additive and multiplicative identities, then we construct a ring R containing S. If S is an ordered semiring with the order $x \leq y$ if and only if $x + a = y$ for some $a \in S$, then S can be identified with the positive cone of the extended ring R. Also the extended ring is regular whenever the semiring S is regular.

1. Introduction and Preliminaries

Semiring is the common generalization of a ring in which the additive reduct does not form a group, but a monoid. Moreover, most of the results developed in semiring theory are borrowed from the theory of groups as well as rings. The set of natural numbers $\mathbb{N}$ with usual addition and multiplication is the typical example for semiring, which can be embedded in the ring of integers. In this paper, by a semiring we mean a nonempty set S with two binary operations $+$ and $\cdot$, called addition and multiplication, such that $(S, +)$ and $(S, \cdot)$ are semigroups, and are connected by ring like distributive property. A semiring S is called additively cancellative[4] if for any $a, b \in S$, $a + c = b + c$ for some $c \in S$ implies $a = b$. The additive (multiplicative) identity, if it exists, is called *zero* (resp. *unity*) and is denoted by 0 (resp. 1). By a semiring we mean a semiring with 0 and 1.

A ring R is *regular* (von Neumann regular), if for every $a \in R$ there exists an $x \in R$ such that $a = axa$. As a natural generalization of this concept to semiring theory, Bourne [1] defined a semiring S to be a *regular in Von Neumann sense*, if for every $a \in S$, there exist some $x, y \in S$ such that $a + axa = aya$. In this paper, we call a semiring S which is regular in the Von Neumann sense simply as *regular semiring.* We follow the norations and terminolagies in [2] and [6] regarding semigroups and [4] and [8] regarding semirings.

2. Extension of semirings to rings

In this section, we primarily identified some sufficient conditions for an additively cancellative and commutative semiring with identity to be a ring.

Proposition 2.1. *Let $(R, +, \cdot)$ be a ring with identity 1 and $\sigma : R \to R$ defined by $\sigma(x) = 1 - x$; then*

(1) *σ is a bijection.*

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(2) $\sigma^2 = I_R$ *, the identity map on R*
(3) *For each idempotent e of* $(R, \cdot), \sigma(e)$ *is also an idempotent*
(4) $e.\sigma(e) = 0 = \sigma(e).e$ *for all multiplicative idempotents e of* $(R, \cdot)$
(5) *If e is a central idempotent of* $(R, \cdot)$*; then so is* $\sigma(e)$
(6) $x + \sigma(x) = 1$ *for all* $x \in R$
(7) $(R, +)$ *is a cancellative semigroup*

Proof. Let $x, y \in R$, such that $\sigma(x) = \sigma(y)$, then $1 - x = 1 - y$ which imply $x = y$. For each $x \in R$, there exist $1 - x \in R$ such that $\sigma(1 - x) = x$, hence σ is onto and hence a bijection. For every $x \in R$, $\sigma^2(x) = \sigma(1 - x) = x = I_R(x)$, this proves (2). Let e be an idempotent of $(R, .)$, it can be easily proved that $\sigma(e)$ is also an idempotent using the ring axioms.

Now assume $e \in E^{\cdot}(R)$, the set of multiplicative idempotents of R, then

$$e.\sigma(e) = e(1 - e) = e - e^2 = 0.$$

Let e be a central idempotent of R; then $e.x = x.e$, for all $x \in R$. We have $[e + \sigma(e)]x = x[e + \sigma(e)]$, which imply $\sigma(e).x = x.\sigma(e)$. Thus $\sigma(e)$ is a central idempotent of R. □

In the following discussion, we consider a semiring $(S, +, .)$ with 0 and 1, where $(S, +)$ is a cancellative and commutative semigroup and we identify one characterization for such a semiring to be a ring.

Proposition 2.2. *Let* $(S, +, \cdot)$ *be a semiring with* 0 *and* 1 *,where* $(S, +)$ *is a cancellative and commutative semigroup and a map* $\sigma : S \to S$ *such that* $x+\sigma(x) = 1$*, for all* $x \in S$*. Then* σ *is a bijection satisfying the conditions 2, 3,4, and 5 of proposition 2.1 .*

Proof. Let $x, y \in S$ be such that $\sigma(x) = \sigma(y)$; Since we have $x + \sigma(x) = y + \sigma(y)$, and by the cancellation property we get $x = y$.

Now for each $y \in S$, we have $y + \sigma(y) = 1$, more over $\sigma(y) + \sigma(\sigma(y)) = 1$. Also by cancellation law, $\sigma(\sigma(y)) = y$ and hence $\sigma(y)$ acts as the pre image of each $y \in S$. Thus σ is a bijection.

To prove (3), consider an idempotent e of $(S, \cdot)$. Then $e + \sigma(e) = 1$. Also, $e(e+\sigma(e)) = e$, which leads to $e\sigma(e) = 0$. Now, $e+\sigma(e) = 1$ implies $[e+\sigma(e)]\sigma(e) = \sigma(e)$. This implies $e\sigma(e) + \sigma(e)\sigma(e) = \sigma(e)$. Therefore $\sigma(e)\sigma(e) = \sigma(e)$, since $e\sigma(e) = 0$. Thus $\sigma(e)$ is also an idempotent.

Property (4) follows from the proof of (3). Towards proving (5), let e be a central idempotent of S, then $xe = ex$ for all $x \in S$.

Now consider $x.\sigma(e)$ for $x \in S$. Since $e + \sigma(e) = 1$, we have $ex + \sigma(e)x = x = xe+x\sigma(e)$. So by cancellation property, $\sigma(e)x = x\sigma(e)$. Hence, $\sigma(e)$ is central. □

Theorem 2.3. *Let* $(S, +, \cdot)$ *be a semiring with* 0 *and* 1*, where* $(S, +)$ *is a cancellative and commutative semigroup and* $\sigma : S \to S$ *be a bijection on S such that* $x + \sigma(x) = 1$*, for all* $x \in S$*, then* $(S, +, \cdot)$ *is a ring.*

Proof. As $(S, +)$ is a semigroup with identity 0, it is sufficient to prove the existence of additive inverse for each element of S. For each $x \in S$, we have $1 + x \in S$. Also by definition of σ we have, $1+x+\sigma(1+x) = 1$. That is $1+x+\sigma(1+x) = 1+0$. This implies $x+\sigma(1+x) = 0$, since $(S,+)$ is cancellative. Hence, $\sigma(1+x)$ is the additive inverse of x. Moreover, this inverse is unique as σ is one-one. By assumption $(S,+)$ is commutative, the other conditions for a ring are satisfied. □

Now we move on to the concept of partially ordered rings and positive cone of a partially ordered ring. Moreover, we discuss some of its relevant properties for our further construction.

Definition 2.4. A ring $(R,+,\cdot)$ together with a partial order relation $\leq$ satisfying

(1) the additive law of monotony
ie; $a < b \Rightarrow a + c < b + c$ for all $a, b, c \in R$

(2) the multiplicative law of monotony
ie; $a < b \Rightarrow ac \leq bc$ for all $a, b \in R$ and all $c \in P$, where P denotes the positive cone of R defined by $P = \{c \in R/c \geq 0\}$

is said to be *partially ordered ring*, simply denoted by *p.o ring* [5].

For such a p.o ring $(R,+,\cdot,\leq)$, the relation $\leq$ is uniquely determined by the positive cone P as follows;

$$a \leq b \Leftrightarrow a + x = b$$

for some $x \in P$. Moreover, P is a subsemiring of R.
On the other hand, each subsemiring P of $(R,+,\cdot)$ satisfying $P \cap -P = \{0\}$, where $-P = \{-p; p \in R\}$ determines a partial order $\leq$ on R such that, $a \leq b$ if and only if $b - a \in P$. Then R is a p.o ring with P as it's positive cone [7].

H J Weinert and U Hebisch [5] established a characterization as semiring embeddable in to a ring if and only if it is additively cancellative and commutative. In the following theorem, we construct an embedding of an additively cancellative and commutative semiring into a ring. A further construction has been studied by Thomas C Craven [3] in the name 'the ring of differences', which resembles with the extension of naturalcancellative numbers to integers .In addition to this, we can define an ordering on the ring such that the positive cone of it coincides with the given semiring.

Theorem 2.5. *Let $(S,+,\cdot)$ be an additively commutative and cancellative semiring with* 0 *and* 1*, then there exist a ring R containing S.*

Proof. The characterization theorem by H J Weinert and U Hebisch [5] ensures the existence of such a ring in which the semiring can be embedded in.

Construction. Let $S \times S = \{(s,t)/s, t \in S\}$ and $R = S \times S/\sim$, where $\sim$ is is defined as follows;
$(a,b) \sim (c,d) \iff a + d = b + c$ in S. It can be proved that this relation is an equivalence relation, for that it is sufficient to prove the transitive property.
Let $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$; then,

$$a + d = b + c \text{ and } c + f = d + e$$
$$\text{Therefore}, a + d + c + f = b + c + d + e$$

By cancellative and commutative assumption, $a + f = b + e$ and hence, $(a, b) \sim (e, f)$.
Now, we define addition and multiplication on R as follows,

$$[(a, b)] + [(c, d)] = [(a + c, b + d)]$$

and

$$[(a, b)].[(c, d)] = [(ac + bd, ad + bc)]$$

Clearly the operation addition is well defined. Now we prove multiplication also well defined.
For that, let $(a_1, b_1) \sim (a_2, b_2)$ and $(c_1, d_1) \sim (c_2, d_2)$
Then for, $i = 1, 2$, we have $[(a_i, b_i)].[(c_i, d_i)] = [(a_ic_i + b_id_i, a_id_i + b_ic_i)]$
Now we claim that, $a_1c_1 + b_1d_1 + a_2d_2 + b_2c_2 = a_2c_2 + b_2d_2 + a_1d_1 + b_1c_1$.
Since $(a_1, b_1) \sim (a_2, b_2)$, we have

$$\begin{aligned} a_1 + b_2 &= b_1 + a_2 \\ a_1c_1 + b_2c_1 &= b_1c_1 + a_2c_1 \\ b_1d_1 + a_2d_1 &= a_1d_1 + b_2d_1 \end{aligned}$$

and $(c_1, d_1) \sim (c_2, d_2) \implies c_1 + d_2 = d_1 + c_2$, which gives the following;

$$\begin{aligned} a_2c_1 + a_2d_2 &= a_2d_1 + a_2c_2 \\ b_2d_1 + b_2c_2 &= b_2c_1 + b_2d_2 \end{aligned}$$

Using the cancellative property, our claim can be proved, and thus we have

$$[(a_1, b_1)].[(c_1, d_1)] = [(a_2, b_2)].[(c_2, d_2)]$$

Hence, the multiplication in R is well defined.
The congruence class of $(0, 0)$, where 0 is the additive identity of S acts as the additive identity for R.
Even if the semiring does not contain the additive identity, we can regard the equivalence class containing the elements of the form (x, x) for $x \in S$ as the additive identity for the ring R.

Now, we can find an element $[(1, 0)]$ in R such that

$$[(a, b)].[(1, 0)] = [(1, 0)].[(a, b)] = [(a, b)]$$

Hence, $[(1, 0)]$ acts as the multiplicative identity in R and this is an additional property for a ring, provided if the semiring posses additive and multiplicative identities.
Commutativity and associativity can be proved easily. Now we prove the existence of additive inverse
For $[(a, b)] \in R$, we can find an equivalence class $[(c, d)]$ such that $[(a, b)] + [(c, d)] = [(0, 0)]$
Since we have $a + b + 0 = b + a + 0$, which imply $(a + b, b + a) \sim (0, 0)$
Hence, $[(a, b)] + [(b, a)] = [(0, 0)]$.

Now it remains to prove the distributive property,
for let, $[(a,b)],[(c,d)],[(e,f)] \in R$

$$\begin{aligned}[(a,b)].\{[(c,d)]+[(e,f)]\} &= [(a,b)].[(c+e,d+f)] \\ &= [(a(c+e)+b(d+f),a(d+f)+b(c+e))] \\ &= [(ac+ae+bd+bf,ad+af+bc+be)]\end{aligned}$$

$$\begin{aligned}[(a,b)].[(c,d)]+[(a,b)].[(e,f)] &= [(ac+bd,ad+bc)]+[(ae+bf,af+be)] \\ &= [(ac+bd+ae+bf,ad+bc+af+be)] \\ &= [(a,b)].\{[(c,d)]+[(e,f)]\}\end{aligned}$$

Now we show that S is isomorphic to a subsemiring of $(R,+,\cdot)$.
Consider $\psi : S \to R$ by $\psi(a) = [(a,0)]$
To prove ψ is one-one, let $[(a,0)] = [(b,0)]$,then $(a,0) \sim (b,0)$ and hence $a = b$. Moreover, in the construction given in Theorem (2.5), we can define an order relation in R, such that $[(x,y)] \geq [(0,0)]$ if and only if $x = t+y$ for some $t \in S$.

$$\begin{aligned}\text{The positive cone of } R &= \{[(x,y)]/[(x,y)] \geq [(0,0)]\} \\ &= \{[(t+y,y)/y \in S\} \\ &= \{[(t,0)]/t \in S\}, \text{ since } (t+y,y) \sim (t,0)\end{aligned}$$

Hence the positive cone of the ring, $P = \{[(x,0)]/x \in S\}$, and we can define a map $\phi : P \to S$, such that $\phi([(x,0)]) = x$, for all $[(x,0)] \in P$ and ϕ can be proved as an isomorphism. Thus,the semiring S becomes the positive cone of the ring. □

In totally ordered semirings, we can realize this as positive cone of rings. We consider partial order on S such that $0 \leq x$ for all $x \in S$ and $x \leq y$ if and only if $y = x+a$ for some $a \in S$.

Theorem 2.6. *Let $(S,+,\cdot)$ be an additively cancellative, commutative totally ordered semiring with the order relation $x \leq y$ if and only if there exist $a \in S$ such that $a+x = y$ and if there exist an element $0 \in S$ such that $x+0 = x$, for some $x \in S$, then 0 will act as additive zero and S can be constructed as the positive cone of a ring.*

Proof. Suppose there be an element $0 \in S$ such that $x+0 = x$,for some $x \in S$ and we prove that this 0 will act as the additive zero for S.
Let $y \in S$ and suppose $x \leq y$,then

$$\begin{aligned}&y = x+a, \text{ for some } a \in S \\ &y+0 = x+a+0 = x+0+a = x+a = y\end{aligned}$$

On the other hand, assume $y \leq x$
Then, $x = y+a$ for some $a \in S$

$$\begin{aligned}x+0 = x &\implies y+a+0 = y+a \\ &\implies y+0+a = y+a \\ &\implies y+0 = y\end{aligned}$$

Hence, the above defined element 0 acts as an identity for addition in S.
Now we are considering a semiring S' which is isomorphic to S. Then there exist a bijection $\sigma : S \to S'$.
We extend the addition in S to the set $R = S \cup S'$ as follows.

$$x + \sigma(x) = 0, \text{for all } x \in S$$

$$\sigma(x) + \sigma(y) = \sigma(x + y) \text{for all } x, y \in S$$

$$x + \sigma(y) = \begin{cases} \sigma(a), & \text{where } x + a = y, \text{if } x \leq y \\ b, & \text{where } y + b = x, \text{if } y \leq x \end{cases}$$

And multiplication in R can be defined as, for $x, y \in S$

$$x\sigma(y) = \sigma(x)y = \sigma(xy),$$

$$\sigma(x)\sigma(y) = xy.$$

Associative property of addition and the distributive property of addition over multiplication in R can be verified by taking different cases. □

Corollary 2.7. *If the semiring S considered in Theorem 2.6 is regular, then the extended ring R is also regular.*

Proof. Let $a \in S'$, the negative cone of R , then $a = \sigma(b)$ for some $b \in S$.
Since $b \in S$, there exist $x, y \in S$ such that $b + bxb = byb$.
Now,

$$\begin{aligned} \sigma(b) + \sigma(b)\sigma(x)\sigma(b) &= \sigma(b) + \sigma(bxb) \\ &= \sigma(b + bxb) \\ &= \sigma(byb) \\ &= \sigma(b)\sigma(y)\sigma(b) \end{aligned}$$

Hence the ring R is regular. □

References

1. Samuel Bourne, *The jacobson radical of a semiring*, Proceedings of the National Academy of Sciences **37** (1951), no. 3, 163–170.
2. Alfred H Clifford and Gordon B Preston, *The algebraic theory of semigroups, vol. 1*, AMS surveys **7** (1961), 1967.
3. Thomas C Craven, *Orderings on semirings*, Semigroup Forum, vol. 43, Springer-Verlag, 1991, pp. 45–52.
4. Jonathan S Golan, *Semirings and their applications*, Springer Science & Business Media, 2013.
5. Udo Hebisch and Hanns Joachim Weinert, *Semirings: algebraic theory and applications in computer science*, vol. 5, World Scientific, 1998.
6. John M Howie, *Fundamentals of semigroup theory*, 12, Oxford University Press, 1995.
7. Jingjing Ma and Warren Wm McGovern, *Division closed partially ordered rings*, Algebra universalis **78** (2017), no. 4, 515–532.
8. F Pastijn and Yuqi Guo, *Semirings which are unions of rings*, Science in China Series A: Mathematics **45** (2002), 172–195.

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