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NON OVERLAPPING FUZZY OCTAGONAL NUMBERS FOR RECOMMENDING THE OPTIMUM EQUITY VALUE OF FINANCIAL SERIES

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Abstract. Risk-neutral fuzzy probability measures play a crucial role in evaluating American fuzzy put option and fuzzy call option pricing models that depend on the amount of increase or decrease of the equity prices. Volatility and risk-free interest rate are the two significant parameters that are affecting the financial market frequently. In 2008, Muzzioli et al [9] and Xcaojian Yu [17] used non-overlapping triangular/trapezoidal and non-overlapping fuzzy trapezoidal numbers respectively to describe the same to study American put option model. In this study, we recommend fuzzy measures involving non-overlapping fuzzy octagonal numbers using which we demonstrate American Put Fuzzy Option Model (APFOM) by incorporating the uncertain values to real in the volatile series.

1. Introduction

Cox et al. [4, 5] binomial tree model was a well-known and renowned tree model in the course of examining option pricing and derivative securities. Muzzioli et al. [12, 11, 10, 13, 14, 15, 9] studied American put option under fuzzy environment by admitting vagueness only in the jump factors. Xcaojian Yu and Zhaozhang Ren [17] have extended the model discussed in [9], by accepting fuzziness in both interest rates with no risk and volatility. Xcaojian Yu et al. [18] have performed empirical research on pricing American put options on 3 - month Euribor futures based on a fuzzy set. Later, Yoshida [20, 22, 24, 23, 21] had been taken the effort to investigate the same. Sumarti et al. [16] studied a dynamic portfolio of American options using a fuzzy binomial method in 2016. Further, several authors worked on a similar situation [19, 2, 3]. For the purpose of complete disclosure, we review the definitions of linear fuzzy octagonal numbers, $\alpha - cut$, arithmetic operations, and their ranking cited from [8] which is essential for further understanding of the present work.

The article is sequenced as given below:
The introduction of risk-neutral fuzzy probability measures involving non-overlapping fuzzy octagonal numbers and their characteristics are discussed in the subsequent sections 2 and 3 respectively. In Section 4, a computational procedure is given to estimate the optimal price of APFOM. The same is validated to substantiate our model cited from *"www.optionistics.com"*. Section 5, presents the conclusion.

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2. Literature Review on Linear fuzzy octagonal numbers

For the purpose of complete disclosure, we review the definitions of linear fuzzy octgonal numbers, cut, arithmetic operations and it's ranking cited from [8] which is essential for further understanding of the present work.

3. Fuzzy probability measures - non-overlapping fuzzy octagonal numbers

This section introduces the no-arbitrage condition to define fuzzy measures involving non-overlapping fuzzy octagonal numbers and execute the American put option model in a fuzzy setup. Let $\widetilde{\mathcal{U}}^{\uparrow} \approx (\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, \mathcal{U}_5, \mathcal{U}_6, \mathcal{U}_7, \mathcal{U}_8; k)$, $\widetilde{\mathcal{D}}^{\downarrow} \approx (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8; k)$. be the up (rise) and down (fall) fuzzy jump factors and $\widetilde{\mathcal{R}}^f \approx (\mathcal{R}_1^f, \mathcal{R}_2^f, \mathcal{R}_3^f, \mathcal{R}_4^f, \mathcal{R}_5^f, \mathcal{R}_6^f, \mathcal{R}_7^f, \mathcal{R}_8^f; k)$ be rate of fuzzy interest at free risk of the underlying fuzzy asset respectively (see Figure ??). Then the $\alpha - cuts$ of $\widetilde{\mathcal{U}}^{\uparrow}$, $\widetilde{\mathcal{D}}^{\downarrow}$ and $\widetilde{\mathcal{R}}^f$ are given by

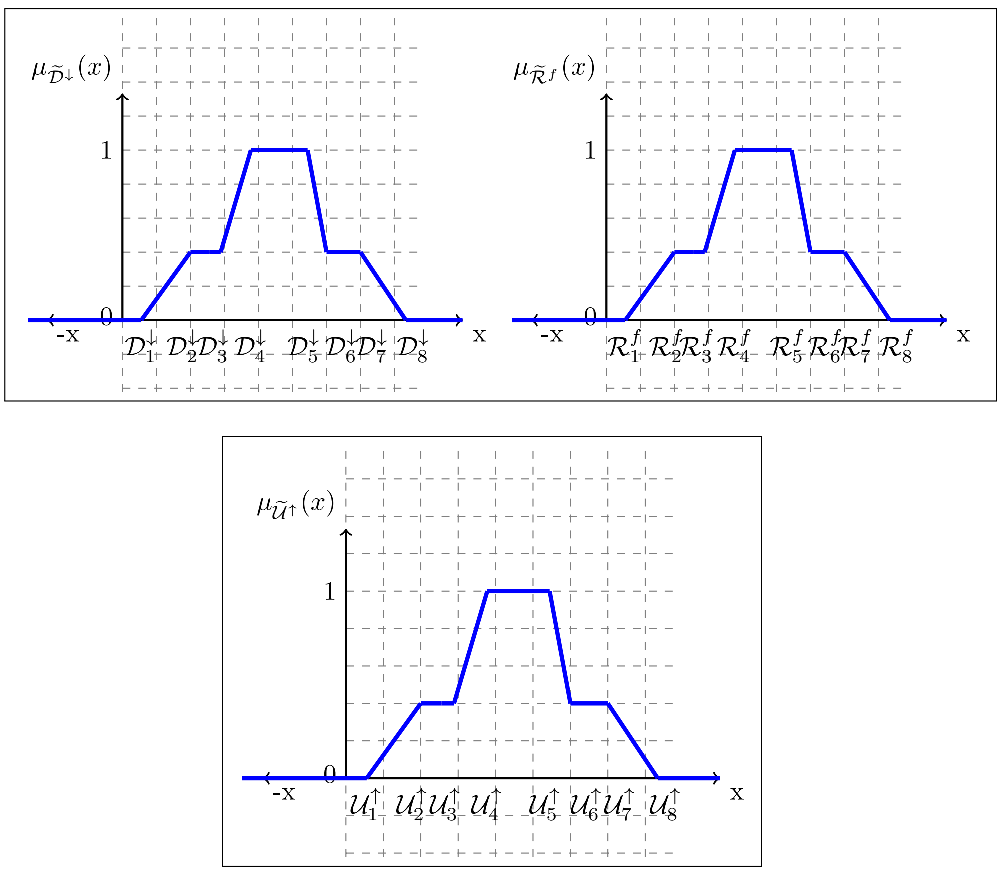

FIGURE 1. No-arbitrage supposition using non-overlapping fuzzy octagonal numbers

$$\widetilde{\mathcal{U}}^{\uparrow}_{\alpha} = \begin{cases} [\underline{\mathcal{U}}^{\uparrow}(\alpha)_1, \overline{\mathcal{U}}^{\uparrow}(\alpha)_1], \text{for} \quad \alpha \in [0, k] \\ [\underline{\mathcal{U}}^{\uparrow}(\alpha)_2, \overline{\mathcal{U}}^{\uparrow}(\alpha)_2], \text{for} \quad \alpha \in (k, 1] \end{cases}$$

$$\widetilde{\mathcal{D}}^{\downarrow}_{\alpha} = \begin{cases} [\underline{\mathcal{D}}^{\downarrow}(\alpha)_1, \overline{\mathcal{D}}^{\downarrow}(\alpha)_1], \text{for} \quad \alpha \in [0, k] \\ [\underline{\mathcal{D}}^{\downarrow}(\alpha)_2, \overline{\mathcal{D}}^{\downarrow}(\alpha)_2], \text{for} \quad \alpha \in (k, 1] \quad \text{and} \end{cases}$$

$$\widetilde{\mathcal{R}}^{f}_{\alpha} = \begin{cases} [\underline{\mathcal{R}}^{f}(\alpha)_1, \overline{\mathcal{R}}^{f}(\alpha)_1], \text{for} \quad \alpha \in [0, k] \\ [\underline{\mathcal{R}}^{f}(\alpha)_2, \overline{\mathcal{R}}^{f}(\alpha)_2], \text{for} \quad \alpha \in (k, 1] \quad \text{where,} \end{cases}$$

$$\underline{\mathcal{U}}^{\uparrow}(\alpha)_1 = \mathcal{U}^{\uparrow}{}_1 + \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2 - \mathcal{U}^{\uparrow}{}_1), \text{ for } \alpha \in [0, k] \tag{3.1}$$

$$\overline{\mathcal{U}}^{\uparrow}(\alpha)_1 = \mathcal{U}^{\uparrow}{}_8 - \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8 - \mathcal{U}^{\uparrow}{}_7), \text{ for } \alpha \in [0, k] \tag{3.2}$$

$$\underline{\mathcal{U}}^{\uparrow}(\alpha)_2 = \mathcal{U}^{\uparrow}{}_3 + \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{U}^{\uparrow}{}_4 - \mathcal{U}^{\uparrow}{}_3), \text{ for } \alpha \in (k, 1] \tag{3.3}$$

$$\overline{\mathcal{U}}^{\uparrow}(\alpha)_2 = \mathcal{U}^{\uparrow}{}_6 - \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5), \text{ for } \alpha \in (k, 1] \tag{3.4}$$

$$\underline{\mathcal{D}}^{\uparrow}(\alpha)_1 = \mathcal{D}^{\uparrow}{}_1 + \frac{\alpha}{k}(\mathcal{D}^{\uparrow}{}_2 - \mathcal{D}^{\uparrow}{}_1) \text{ for } \alpha \in [0, k] \tag{3.5}$$

$$\overline{\mathcal{D}}^{\uparrow}(\alpha)_1 = \mathcal{D}^{\uparrow}{}_8 - \frac{\alpha}{k}(\mathcal{D}^{\uparrow}{}_8 - \mathcal{D}^{\uparrow}{}_7) \text{ for } \alpha \in [0, k] \tag{3.6}$$

$$\underline{\mathcal{D}}^{\uparrow}(\alpha)_2 = \mathcal{D}^{\uparrow}{}_3 + \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{D}^{\uparrow}{}_4 - \mathcal{D}^{\uparrow}{}_3) \text{ for } \alpha \in (k, 1] \tag{3.7}$$

$$\overline{\mathcal{D}}^{\uparrow}(\alpha)_2 = \mathcal{D}^{\uparrow}{}_6 - \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{D}^{\uparrow}{}_6 - \mathcal{D}^{\uparrow}{}_5) \text{ for } \alpha \in (k, 1] \tag{3.8}$$

$$\underline{\mathcal{R}}^{f}(\alpha)_1 = \mathcal{R}^{f}{}_1 + \frac{\alpha}{k}(\mathcal{R}^{f}{}_2 - \mathcal{R}^{f}{}_1), \text{ for } \alpha \in [0, k] \tag{3.9}$$

$$\overline{\mathcal{R}}^{f}(\alpha)_1 = \mathcal{R}^{f}{}_8 - \frac{\alpha}{k}(\mathcal{R}^{f}{}_8 - \mathcal{R}^{f}{}_7), \text{ for } \alpha \in [0, k] \tag{3.10}$$

$$\underline{\mathcal{R}}^{f}(\alpha)_2 = \mathcal{R}^{f}{}_3 + \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{R}^{f}{}_4 - \mathcal{R}^{f}{}_3), \text{ for } \alpha \in (k, 1] \tag{3.11}$$

$$\overline{\mathcal{R}}^{f}(\alpha)_2 = \mathcal{R}^{f}{}_6 - \left(\frac{\alpha - k}{1 - k}\right)(\mathcal{R}^{f}{}_6 - \mathcal{R}^{f}{}_5), \text{ for } \alpha \in (k, 1] \tag{3.12}$$

$\mathcal{P}_{\mathcal{U}^{\uparrow}}(\alpha)$ and $\mathcal{P}_{\mathcal{D}^{\downarrow}}(\alpha)$ respectively represents the $\alpha-cuts$ of the two bounds of the intervals of risk-neutral fuzzy probability measures $\widetilde{\mathcal{P}}_{\mathcal{U}^{\uparrow}}$ and $\widetilde{\mathcal{P}}_{\mathcal{D}^{\downarrow}}$.

$$\mathcal{P}_{\mathcal{U}^{\uparrow}}(\alpha) \quad = \quad \begin{cases} [\underline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha), \overline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha)], \text{for} \quad \alpha \in [0,k] \\ [\underline{p}^{(2)}_{\mathcal{U}^{\uparrow}}(\alpha), \overline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}(\alpha)], \text{for} \quad \alpha \in (k,1] \end{cases} \tag{3.13}$$

$$\mathcal{P}_{\mathcal{D}^{\downarrow}}(\alpha) \quad = \quad \begin{cases} [\underline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha), \overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha)], \text{for} \quad \alpha \in [0,k] \\ [\underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}(\alpha), \overline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}(\alpha)], \text{for} \quad \alpha \in (k,1] \end{cases} \quad \text{and} \tag{3.14}$$

Using equations given from (3.1) to (3.14), we have

$$\underline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha) = \frac{(1+(\mathcal{R}^f{}_1 + \frac{\alpha}{k}(\mathcal{R}^f{}_2 - \mathcal{R}^f{}_1)) - \mathcal{D}^{\downarrow}{}_8 + \frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_8 - \mathcal{D}^{\downarrow}{}_7)}{(\mathcal{U}^{\uparrow}{}_8 - \mathcal{D}^{\downarrow}{}_8) - \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8 - \mathcal{U}^{\uparrow}{}_7 - \mathcal{D}^{\downarrow}{}_8 + \mathcal{D}^{\downarrow}{}_7)}, \alpha \in [0,k] \tag{3.15}$$

$$\overline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha) = \frac{(1+(\mathcal{R}^f{}_8 - \frac{\alpha}{k}(\mathcal{R}^f{}_8 - \mathcal{R}^f{}_7)) - \mathcal{D}^{\downarrow}{}_1 - \frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_2 - \mathcal{D}^{\downarrow}{}_1)}{(\mathcal{U}^{\uparrow}{}_1 - \mathcal{D}^{\downarrow}{}_1) + \frac{\alpha}{k}(\mathcal{U}^{\uparrow}_2 - \mathcal{U}^{\uparrow}{}_1 - \mathcal{D}^{\downarrow}{}_2 + \mathcal{D}^{\downarrow}{}_1)}, \alpha \in [0,k] \tag{3.16}$$

$$\underline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}(\alpha) = \frac{(1+(\mathcal{R}^f{}_3 + (\frac{\alpha-k}{1-k})(\mathcal{R}^f{}_4 - \mathcal{R}^f{}_3)) - \mathcal{D}^{\downarrow}{}_6 + (\frac{\alpha-k}{1-k})(\mathcal{D}^{\downarrow}{}_6 - \mathcal{D}^{\downarrow}{}_5)}{(\mathcal{U}^{\uparrow}{}_6 - \mathcal{D}^{\downarrow}{}_6) - (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5 - \mathcal{D}^{\downarrow}{}_6 + \mathcal{D}^{\downarrow}{}_5)}, \alpha \in (k,1] \tag{3.17}$$

$$\overline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}(\alpha) = \frac{(1+(\mathcal{R}^f{}_6 - (\frac{\alpha-k}{1-k})(\mathcal{R}^f{}_6 - \mathcal{R}^f{}_5)) - \mathcal{D}^{\downarrow}{}_3 - (\frac{\alpha-k}{1-k})(\mathcal{D}^{\downarrow}{}_4 - \mathcal{D}^{\downarrow}{}_3)}{(\mathcal{U}^{\uparrow}{}_3 - \mathcal{D}^{\downarrow}{}_3) + (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_4 - \mathcal{U}^{\uparrow}{}_3 - \mathcal{D}^{\downarrow}{}_4 + \mathcal{D}^{\downarrow}{}_3)}, \alpha \in (k,1] \tag{3.18}$$

$$\underline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha) = \frac{\mathcal{U}^{\uparrow}{}_1 - (1+(\mathcal{R}^f{}_8 - \frac{\alpha}{k}(\mathcal{R}^f{}_8 - \mathcal{R}^f{}_7)) + \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2 - \mathcal{U}^{\uparrow}{}_1)}{(\mathcal{U}^{\uparrow}{}_1 - \mathcal{D}^{\downarrow}{}_1) + \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2 - \mathcal{U}^{\uparrow}{}_1 - \mathcal{D}^{\downarrow}{}_2 + \mathcal{D}^{\downarrow}{}_1)}, \alpha \in [0,k] \tag{3.19}$$

$$\overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha) = \frac{\mathcal{U}^{\uparrow}{}_8 - (1+(\mathcal{R}^f{}_1 + \frac{\alpha}{k}(\mathcal{R}^f{}_2 - \mathcal{R}^f{}_1)) - \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8 - \mathcal{U}^{\uparrow}{}_7)}{(\mathcal{U}^{\uparrow}{}_8 - \mathcal{D}^{\downarrow}{}_8) - \frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8 - \mathcal{U}^{\uparrow}{}_7 - \mathcal{D}^{\downarrow}{}_8 + \mathcal{D}^{\downarrow}{}_7)}, \alpha \in [0,k] \tag{3.20}$$

$$\underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}} = \frac{\mathcal{U}^{\uparrow}{}_3 - (1+(\mathcal{R}^f{}_6 - (\frac{\alpha-k}{1-k})(\mathcal{R}^f{}_6 - \mathcal{R}^f{}_5)) + (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_4 - \mathcal{U}^{\uparrow}{}_3)}{(\mathcal{U}^{\uparrow}{}_3 - \mathcal{D}^{\downarrow}{}_3) + (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_4 - \mathcal{U}^{\uparrow}{}_3 - \mathcal{D}^{\downarrow}{}_4 + \mathcal{D}^{\downarrow}{}_3)}, \alpha \in (k,1] \tag{3.21}$$

$$\overline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}(\alpha) = \frac{\mathcal{U}^{\uparrow}{}_6 - (1+(\mathcal{R}^f{}_3 + (\frac{\alpha-k}{1-k})(\mathcal{R}^f{}_4 - \mathcal{R}^f{}_3)) - (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5)}{(\mathcal{U}^{\uparrow}{}_6 - \mathcal{D}^{\downarrow}{}_6) - (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5 - \mathcal{D}^{\downarrow}{}_6 + \mathcal{D}^{\downarrow}{}_5)}, \alpha \in (k,1] \tag{3.22}$$

Taking $\alpha = 0$, $\alpha = k$ in equations (3.15), (3.16), (3.17), (3.18) and $\alpha = k$, $\alpha = 1$ in (3.19), (3.20), (3.21), (3.22), the two exterior points and the interior points of the risk-neutral fuzzy probability measures that depend on the trends of $\widetilde{\mathcal{U}}^{\uparrow}$, $\widetilde{\mathcal{D}}^{\downarrow}$ and $\widetilde{\mathcal{R}}^f$ are defined by

$$\begin{aligned} \widetilde{\mathcal{P}}_{\mathcal{U}^{\uparrow}} \quad \approx \quad & \Bigg(\frac{1+\mathcal{R}_1 - \mathcal{D}^{\downarrow}{}_8}{\mathcal{U}^{\uparrow}{}_8 - \mathcal{D}^{\downarrow}{}_8}, \frac{1+\mathcal{R}_2 - \mathcal{D}^{\downarrow}{}_7}{\mathcal{U}^{\uparrow}{}_7 - \mathcal{D}^{\downarrow}{}_7}, \frac{1+\mathcal{R}_3 - \mathcal{D}^{\downarrow}{}_6}{\mathcal{U}^{\uparrow}{}_6 - \mathcal{D}^{\downarrow}{}_6}, \frac{1+\mathcal{R}_4 - \mathcal{D}^{\downarrow}{}_5}{\mathcal{U}^{\uparrow}{}_5 - \mathcal{D}^{\downarrow}{}_5}, \\ & \frac{1+\mathcal{R}_5 - \mathcal{D}^{\downarrow}{}_4}{\mathcal{U}^{\uparrow}{}_4 - \mathcal{D}^{\downarrow}{}_4}, \frac{1+\mathcal{R}_6 - \mathcal{D}^{\downarrow}{}_3}{\mathcal{U}^{\uparrow}{}_3 - \mathcal{D}^{\downarrow}{}_3}, \frac{1+\mathcal{R}_7 - \mathcal{D}^{\downarrow}{}_2}{\mathcal{U}^{\uparrow}{}_2 - \mathcal{D}^{\downarrow}{}_2}, \frac{1+\mathcal{R}_8 - \mathcal{D}^{\downarrow}{}_1}{\mathcal{U}^{\uparrow}{}_1 - \mathcal{D}^{\downarrow}{}_1}; k \Bigg) \end{aligned} \tag{3.23}$$

$$\widetilde{\mathcal{P}}_{\mathcal{D}^{\downarrow}} \approx \Big(\frac{\mathcal{U}^{\uparrow}{}_1-(1+\mathcal{R}_8)}{\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_1},\frac{\mathcal{U}^{\uparrow}{}_2-(1+\mathcal{R}_7)}{\mathcal{U}^{\uparrow}{}_2-\mathcal{D}^{\downarrow}{}_2},\frac{\mathcal{U}^{\uparrow}{}_3-(1+\mathcal{R}_6)}{\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_3},\frac{\mathcal{U}^{\uparrow}{}_4-(1+\mathcal{R}_5)}{\mathcal{U}^{\uparrow}{}_4-\mathcal{D}^{\downarrow}{}_4}, \tag{3.24}$$
$$\frac{\mathcal{U}^{\uparrow}{}_5-(1+\mathcal{R}_4)}{\mathcal{U}^{\uparrow}{}_5-\mathcal{D}^{\downarrow}{}_5},\frac{\mathcal{U}^{\uparrow}{}_6-(1+\mathcal{R}_3)}{\mathcal{U}^{\uparrow}{}_6-\mathcal{D}^{\downarrow}{}_6},\frac{\mathcal{U}^{\uparrow}{}_7-(1+\mathcal{R}_2)}{\mathcal{U}^{\uparrow}{}_7-\mathcal{D}^{\downarrow}{}_7},\frac{\mathcal{U}^{\uparrow}{}_8-(1+\mathcal{R}_1)}{\mathcal{U}^{\uparrow}{}_8-\mathcal{D}^{\downarrow}{}_8};k\Big)$$

From this, we analyze that dualistic circumstances are true.

4. Evaluation Method of APFOM on fuzzy future contract

We, now examine how risk-neutral probability measures behave when it is represented by non-overlapping fuzzy octagonal numbers using the presumptions considered here. Also, we record a computational procedure to obtain optimal prices of fuzzy put options based on the American style.

Characteristics of risk-neutral fuzzy probabilities using non-overlapping fuzzy octagonal numbers:

Here, we assume that the following no arbitrage inequality holds: $\mathcal{D}^{\downarrow}{}_i(1+\mathcal{R}^f{}_i)\mathcal{U}^{\uparrow}{}_i$ and $0\ \mathcal{P}_{\mathcal{U}^{\uparrow}{}_i}, \mathcal{P}_{\mathcal{D}^{\downarrow}{}_i} 1, i=1,2,\dots,8$

$$\begin{aligned}
\underline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha) &= \frac{(1+\mathcal{R}^f{}_1)-\mathcal{D}^{\downarrow}{}_8+\frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_8-\mathcal{D}^{\downarrow}{}_7)}{(\mathcal{U}^{\uparrow}{}_8-\mathcal{D}^{\downarrow}{}_8)-\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7-\mathcal{D}^{\downarrow}{}_8+\mathcal{D}^{\downarrow}{}_7)} \quad \text{for} \quad \alpha\in[0,k] \\
&= \frac{N_1}{D_1} \\
\underline{\mathcal{P}}^{(1)\prime}_{\mathcal{U}^{\uparrow}}(\alpha) &= \frac{(D_1-N_1)(\mathcal{D}^{\downarrow}{}_8-\mathcal{D}^{\downarrow}{}_7)+N^2(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7)}{kD1^2}
\end{aligned}$$

$$\text{Also,} \quad \underline{\mathcal{P}}^{(1)\prime\prime}_{\mathcal{U}^{\uparrow}}(\alpha) = \underline{\mathcal{P}}^{(1)\prime}_{\mathcal{U}^{\uparrow}}(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7-\mathcal{D}^{\downarrow}{}_8+\mathcal{D}^{\downarrow}{}_7)}{kD_1}$$

Similarly, we have

$$\begin{aligned}
\overline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}(\alpha) &= \frac{(1+\mathcal{R}^f{}_8)-\mathcal{D}^{\downarrow}{}_1-\frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_2-\mathcal{D}^{\downarrow}{}_1)}{(\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_1)+\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_2+\mathcal{D}^{\downarrow}{}_1)} \\
&= \frac{N_2}{D_2} \\
\overline{\mathcal{P}}^{(1)\prime}_{\mathcal{U}}(\alpha) &= -\frac{(D_2-N_2)(\mathcal{D}^{\downarrow}{}_2-\mathcal{D}^{\downarrow}{}_1)-N_2(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1)}{kD_2^2}
\end{aligned}$$

$$\overline{\mathcal{P}}^{(1)\prime\prime}_{\mathcal{U}^{\uparrow}}(\alpha) = -\underline{\mathcal{P}}^{(1)\prime}_{\mathcal{U}^{\uparrow}}(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_2+\mathcal{D}^{\downarrow}{}_1)}{kD_2}$$

$$\begin{aligned}
\underline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha) &= \frac{\mathcal{U}^{\uparrow}{}_1-(1+\mathcal{R}^f{}_8)+\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1)(\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_1)+\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_2+\mathcal{D}^{\downarrow}{}_1)}{} \\
&= \frac{N_5}{D_5} \\
\underline{\mathcal{P}}^{(1)\prime}_{\mathcal{D}^{\downarrow}}(\alpha) &= \frac{(D_5-N_5)(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1)+N_3(\mathcal{D}^{\downarrow}{}_2-\mathcal{D}^{\downarrow}{}_1)}{kD_5^2}
\end{aligned}$$

$$\text{Also,}\quad \underline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}{}''(\alpha) = -\underline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_2-\mathcal{U}^{\uparrow}{}_1-\mathcal{D}^{\downarrow}{}_2+\mathcal{D}^{\downarrow}{}_1)}{kD_5}$$

In a similar manner,

$$\begin{aligned}\overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}(\alpha) &= \frac{\mathcal{U}^{\uparrow}{}_8-(1+\mathcal{R}^f{}_1)-\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}_7)}{(\mathcal{U}^{\uparrow}{}_8-\mathcal{D}^{\downarrow}{}_8)-\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7-\mathcal{D}^{\downarrow}{}_8+\mathcal{D}^{\downarrow}{}_7)}\\ &= \frac{N_6}{D_6}\\ \overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}{}'(\alpha) &= -\frac{(D_6-N_6)(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7)-N_4(\mathcal{D}^{\downarrow}{}_8-\mathcal{D}^{\downarrow}{}_7)}{kD_6^2}\end{aligned}$$

$$\text{Also,}\quad \overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}{}''(\alpha) = -\overline{\mathcal{P}}^{(1)}_{\mathcal{D}^{\downarrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_8-\mathcal{U}^{\uparrow}{}_7-\mathcal{D}^{\downarrow}{}_8+\mathcal{D}^{\downarrow}{}_7)}{kD_6}$$

$$\begin{aligned}\underline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}(\alpha) &= \frac{(1+\mathcal{R}^f{}_3)-\mathcal{D}^{\downarrow}{}_6+\frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_6-\mathcal{D}^{\downarrow}{}_5)}{(\mathcal{U}^{\uparrow}{}_6-\mathcal{D}^{\downarrow}{}_6)-\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_6-\mathcal{U}^{\uparrow}{}_5-\mathcal{D}^{\downarrow}{}_6+\mathcal{D}^{\downarrow}{}_5)} \quad \text{for}\quad \alpha\in(k,1]\\ &= \frac{N_1}{D_1}\\ \underline{\mathcal{P}}^{(2)}_{\mathcal{U}}{}'(\alpha) &= \frac{(D_1-N_1)(\mathcal{D}^{\downarrow}{}_6-\mathcal{D}^{\downarrow}{}_5)+N_2(\mathcal{U}^{\uparrow}{}_6-\mathcal{U}^{\uparrow}{}_5)}{kD_1^2}\end{aligned}$$

$$\text{Also,}\quad \underline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}{}''(\alpha) = \underline{\mathcal{P}}^{(1)}_{\mathcal{U}^{\uparrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_6-\mathcal{U}^{\uparrow}{}_5-\mathcal{D}^{\downarrow}{}_6+\mathcal{D}^{\downarrow}{}_5)}{kD_1}$$

$$\begin{aligned}\overline{\mathcal{P}}^{(2)}_{\mathcal{U}}(\alpha) &= \frac{(1+\mathcal{R}^f{}_6)-\mathcal{D}^{\downarrow}{}_3-\frac{\alpha}{k}(\mathcal{D}^{\downarrow}{}_4-\mathcal{D}^{\downarrow}{}_3)}{(\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_3)+\frac{\alpha}{k}(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_4+\mathcal{D}^{\downarrow}{}_3)}\\ &= \frac{N_2}{D_2}\\ \overline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}{}'(\alpha) &= -\frac{(D_2-N_2)(\mathcal{D}^{\downarrow}{}_4-\mathcal{D}^{\downarrow}{}_3)-N_2(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3)}{kD_2^2}\end{aligned}$$

$$\overline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}{}''(\alpha) = -\overline{\mathcal{P}}^{(2)}_{\mathcal{U}^{\uparrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_4+\mathcal{D}^{\downarrow}{}_3)}{kD_2}$$

Now,

$$\begin{aligned}\underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}(\alpha) &= \frac{\mathcal{U}^{\uparrow}{}_3-(1+\mathcal{R}^f{}_6)+(\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3)}{(\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_3)+(\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_4+\mathcal{D}^{\downarrow}{}_3)}\\ &= \frac{N_7}{D_7}\\ \underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}{}'(\alpha) &= \frac{(D_7-N_7)(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3)+N_7(\mathcal{D}^{\downarrow}{}_4-\mathcal{D}^{\downarrow}{}_3)}{(1-k)D_7^2}\end{aligned}$$

$$\underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}{}''(\alpha) = \underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_4-\mathcal{U}^{\uparrow}{}_3-\mathcal{D}^{\downarrow}{}_4+\mathcal{D}^{\downarrow}{}_3)}{(1-k)D_7}$$

In line with the above,

$$\begin{aligned}\overline{\mathcal{P}}^{(2)}_{\mathcal{D}}(\alpha) &= \frac{\mathcal{U}^{\uparrow}{}_6 - (1+\mathcal{R}^f{}_3) - (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5)}{(\mathcal{U}^{\uparrow}{}_6 - \mathcal{D}^{\downarrow}{}_6) - (\frac{\alpha-k}{1-k})(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5 - \mathcal{D}^{\downarrow}{}_6 + \mathcal{D}^{\downarrow}{}_5)} \\ &= \frac{N_8}{D_8} \\ \overline{\mathcal{P}}^{(2)}_{d}{}'(\alpha) &= -\frac{(D_8 - N_8)(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5) - N_8(\mathcal{D}^{\downarrow}{}_6 - \mathcal{D}^{\downarrow}{}_5)}{(1-k)D_8^2}\end{aligned}$$

$$\overline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}{}''(\alpha) = -\underline{\mathcal{P}}^{(2)}_{\mathcal{D}^{\downarrow}}{}'(\alpha)2\frac{(\mathcal{U}^{\uparrow}{}_6 - \mathcal{U}^{\uparrow}{}_5 - \mathcal{D}^{\downarrow}{}_6 + \mathcal{D}^{\downarrow}{}_5)}{(1-k)D_8}$$

We realize that the second derivative of the risk-neutral probabilities with respect to α is positive and negative for both left and right bounds respectively. Further, if the difference in up jump factors is equal to the difference in down jump factors, then the same is linear in α. Also, the left and right bounds of the non-overlapping octagonal fuzzy number that characterize the fuzzy risk-neutral probabilities are convex and concave respectively. In case, if $\alpha = 1$, these bounds will converge at one particular point. As a result of this, the market is complete and a unique risk-neutral fuzzy probability measure will exist.

Definition 4.1. Let the fuzzy future contract be expired at a maturity period, T. Then, uncertain future reward at time t is $\widetilde{\mathcal{F}}_{t,i}$ defined as,

$$\widetilde{\mathcal{F}}_{t,i} \approx (\widetilde{1} + \widetilde{\mathcal{R}^f})^{T-t}\widetilde{S}_{t,i},\ i = 0,1,\ldots,t \text{ and } t = 0,1,2,\ldots,T \tag{4.1}$$

here,

$$\widetilde{S}_{t,i} \approx \widetilde{S}_0(\bar{\mathcal{U}^{\uparrow}})(\bar{\mathcal{D}^{\downarrow}}^{t-i}) \tag{4.2}$$

is the fuzzy stock price. During the expiration date of the contract at $t = T$, fuzzy future price $\widetilde{\mathcal{F}}_{T,i}$ coincides with the fuzzy stock price $\widetilde{S}_{T,i}$.

Definition 4.2. The *snell envelope* of the American fuzzy pay off process $(\widetilde{p}^{\mathcal{A}}_t)_{t\in T}$ where $\widetilde{p}^{\mathcal{A}}_t(\widetilde{\mathcal{F}}_{t,i}) \approx \widetilde{K} - \widetilde{\mathcal{F}}_{t,i},\ t = 0,1,2,\ldots,T,\ i = 0,1,2,\ldots,t$ is an $\mathcal{M}_t-$ adapted American fuzzy put price process $(\widetilde{\mathcal{V}}^{\mathcal{A}}_t)_{t\in T}$ defined by

$$\widetilde{\mathcal{V}}^{\mathcal{A}}_T(\widetilde{\mathcal{F}}_{T,i}) \approx \max\left\{\widetilde{p}^{\mathcal{A}}_T(\widetilde{\mathcal{F}}_{T,i}), \widetilde{0}\right\}$$

and

$$\widetilde{\mathcal{V}}^{\mathcal{A}}_t(\widetilde{\mathcal{F}}_{t,i}) \approx \max\left\{\widetilde{p}^{\mathcal{A}}_t(\widetilde{\mathcal{F}}_{t,i}), \frac{\widetilde{1}}{\widetilde{1} + \widetilde{\mathcal{R}}^f}\widetilde{E}_t(\widetilde{\mathcal{V}}^{\mathcal{A}}_{t+1}(\widetilde{\mathcal{F}}_{t+1,i}))\right\}$$

where $\widetilde{E}_t(\widetilde{\mathcal{V}}^{\mathcal{A}}_{t+1}(\widetilde{\mathcal{F}}_{t+1,i})) \approx (\widetilde{\mathcal{P}}_{\mathcal{U}}\widetilde{\mathcal{V}}^{\mathcal{A}}_{t+1}(\widetilde{\mathcal{F}}_{t+1,i}) + \widetilde{\mathcal{P}}_{\mathcal{D}}\widetilde{\mathcal{V}}^{\mathcal{A}}_{t+1}(\widetilde{\mathcal{F}}_{t+1,i}))$ for $i = 0,1,\ldots,t,\ t = T-1, T-2, \ldots 0$.

5. Aim

To obtain APFOM prices using non-overlapping fuzzy octagonal numbers.
Design:

Calculate the fuzzy jump factors $\mathcal{U}\uparrow_1 = e^{\sigma_1\sqrt{\Delta t}}, \mathcal{U}\uparrow_2 = e^{\sigma_2\sqrt{\Delta t}}, \mathcal{U}\uparrow_3 = e^{\sigma_3\sqrt{\Delta t}}, \mathcal{U}\uparrow_4 = e^{\sigma_4\sqrt{\Delta t}}, \mathcal{U}\uparrow_5 = e^{\sigma_5\sqrt{\Delta t}},$ $\mathcal{U}\uparrow_6 = e^{\sigma_6\sqrt{\Delta t}}, \mathcal{U}\uparrow_7 = e^{\sigma_7\sqrt{\Delta t}}, \mathcal{U}\uparrow_8 = e^{\sigma_8\sqrt{\Delta t}}$, where $\Delta t = T/n$ and $\mathcal{D}\downarrow_1 = \frac{1}{\mathcal{U}\uparrow_8}, \mathcal{D}\downarrow_2 = \frac{1}{\mathcal{U}\uparrow_7}, \mathcal{D}\downarrow_3 = \frac{1}{\mathcal{U}\uparrow_6}, \mathcal{D}\downarrow_4 = \frac{1}{\mathcal{U}\uparrow_5}, \mathcal{D}_5 = \frac{1}{\mathcal{U}\uparrow_4},$ $\mathcal{D}\downarrow_6 = \frac{1}{\mathcal{U}\uparrow_3}, \mathcal{D}\downarrow_7 = \frac{1}{\mathcal{U}\uparrow_2}, \mathcal{D}\downarrow_8 = \frac{1}{\mathcal{U}\uparrow_1}.$
Compute the fuzzy stock prices using equation (4.2).
Determine $\widetilde{\mathcal{F}}_t$ for the given fuzzy stock prices using equation (4.1).
Calculate the risk-neutral fuzzy probabilities using equations (3.23) and (3.24).
Obtain American put fuzzy option prices using Definition 4.2.

6. Results and Discussions

We use the following Microsoft stock options data (see Table 1) recorded from *optionistics* website on 04/01/2020.We develop (MATLAB) programs for the computational procedures discussed in this paper
The one-period interest rate is 1.99. Expiration date T= 29/360.
Risk-free fuzzy interest rate is $\widetilde{\mathcal{R}}^f \approx [1.89, 1.92, 1.94, 1.99, 2.02, 2.04, 2.07, 2.09]$.
Fuzzy volatility $\widetilde{\sigma} \approx (0.206, 0.209, 0.212, 0.217, 0.220, 0.223, 0.226, 0.228)$.
Consider $t = 2$, the number of steps in a fuzzy binomial tree.

MSFT	Specifications
Spot price	87.11
Fixed price	88.00
IV	0.217
Maturity Date	2 Feb18
Option style	Put
ROI	1.99

TABLE 1. Setting of the study - Option Spread

By step (1), the fuzzy jump factors are calculated as
$\widetilde{\mathcal{U}\uparrow} \approx [1.04221, 1.04284, 1.04347, 1.04451, 1.04514, 1.04577, 1.04640, 1.04682]$
$\widetilde{\mathcal{D}\downarrow} \approx [0.95527, 0.95566, 0.95623, 0.95681, 0.95739, 0.95834, 0.95892, 0.95950]$.

By step (2), we get the following fuzzy stock prices (see Figure 2):

By using step (3), we obtain fuzzy future prices as shown below: (see Figure 3)
$\widetilde{\mathcal{F}}_0 \approx (87.2427, 87.2448, 87.2462, 87.2497, 87.2518, 87.2532, 87.2553, 87.2567)$
$\widetilde{\mathcal{F}}_1^{\mathcal{U}\uparrow} \approx (90.8560, 90.9121, 90.9677, 91.0602, 91.1162, 91.1719, 91.2279, 91.2653)$
$\widetilde{\mathcal{F}}_1^{\mathcal{D}\downarrow} \approx (83.2769, 83.3119, 83.3623, 83.4145, 83.4661, 83.5496, 83.6011, 83.6524)$

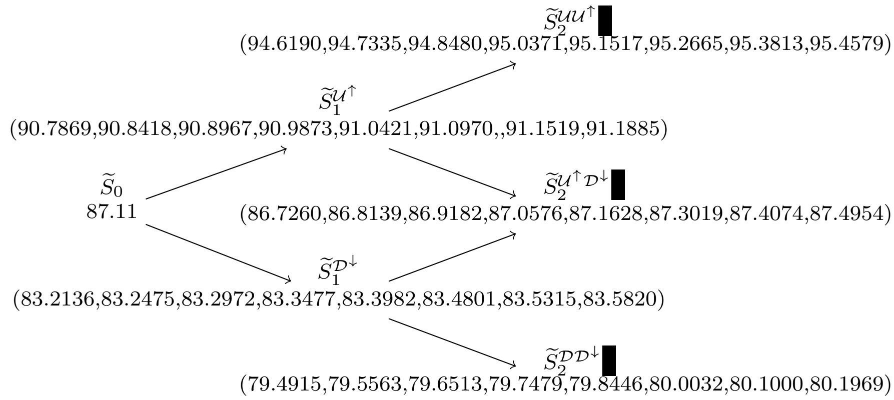

FIGURE 2. Fuzzy binary share prices tree

$\widetilde{\mathcal{F}}_2^{\mathcal{UU}\uparrow} \approx (94.6190, 94.7335, 94.8480, 95.0371, 95.1517, 95.2665, 95.3813, 95.4579)$
$\widetilde{\mathcal{F}}_2^{\mathcal{U}\uparrow\mathcal{D}\downarrow} \approx (86.7260, 86.8139, 86.9182, 87.0576, 87.1628, 87.3019, 87.4074, 87.4954)$
$\widetilde{\mathcal{F}}_2^{\mathcal{DD}\downarrow} \approx (79.4915, 79.5563, 79.6513, 79.7479, 79.8446, 80.0032, 80.1000, 80.1969)$

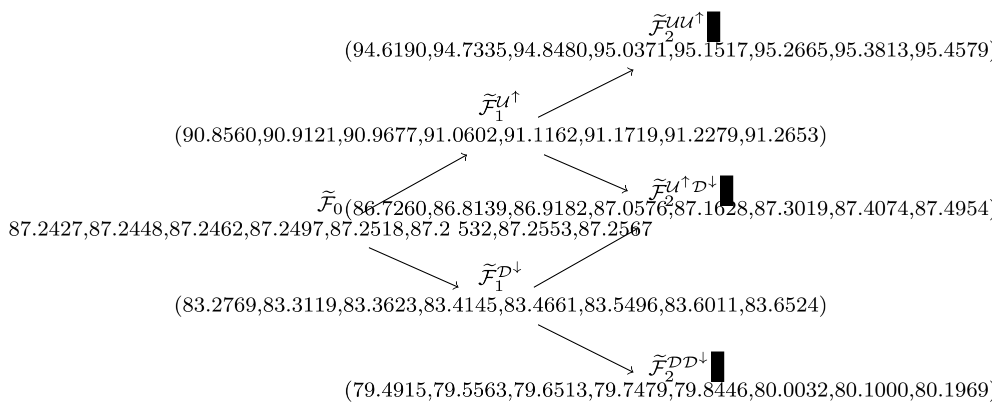

FIGURE 3. Fuzzy binary future prices tree

By step (4), we estimate risk-neutral fuzzy probabilities
$\widetilde{\mathcal{P}}_{\mathcal{U}\uparrow} \approx [0.4725, 0.4784, 0.4854, 0.4947, 0.5018, 0.5111, 0.5182, 0.5242]$
$\widetilde{\mathcal{P}_{\mathcal{D}}^{\downarrow}} \approx [0.4758, 0.4818, 0.4889, 0.4982, 0.5053, 0.5146, 0.5216, 0.5275]$

By step (5), we evaluate the American put fuzzy option prices as shown below:

$$\begin{aligned}
\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}\mathcal{U}^{\uparrow}}) &\approx \max\{\widetilde{K} - \widetilde{\mathcal{F}}_2^{\mathcal{U}\mathcal{U}^{\uparrow}}, \widetilde{0}\} \approx (0,0,0,0,0,0,0,0)\\
\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}}) &\approx \max\{\widetilde{K} - \widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}}, \widetilde{0}\}\\
&\approx (0.5046, 0.5926, 0.6981, 0.8372, 0.9424, 1.0818, 1.1861, 1.274)\\
\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{D}\mathcal{D}^{\downarrow}}) &\approx \max\{\widetilde{K} - \widetilde{\mathcal{F}}_2^{\mathcal{D}\mathcal{D}^{\downarrow}}, \widetilde{0}\}\\
&\approx (7.8031, 7.9, 7.9968, 8.1554, 8.2521, 8.3487, 8.4437, 8.5085)\\
\widetilde{\mathcal{V}}_1^{\mathcal{A}}(\widetilde{\mathcal{F}}_1^{\mathcal{U}^{\uparrow}}) &\approx \max\left\{\widetilde{K} - \widetilde{\mathcal{F}}_1^{\mathcal{U}}, \frac{\widetilde{1}}{(\widetilde{1} + \widetilde{\mathcal{R}^f})}(\widetilde{\mathcal{P}}_{\mathcal{U}^{\uparrow}}\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}\mathcal{U}^{\uparrow}}) + \widetilde{\mathcal{P}}_{\mathcal{D}^{\downarrow}}\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}}))\right\}\\
&\approx (0.2399, 0.2853, 0.3410, 0.4168, 0.4758, 0.5563, 0.6182, 0.6715)\\
\widetilde{\mathcal{V}}_1^{\mathcal{A}}(\widetilde{\mathcal{F}}_1^{\mathcal{D}^{\downarrow}}) &\approx \max\left\{\widetilde{K} - \widetilde{\mathcal{F}}_1^{\mathcal{D}^{\downarrow}}, \frac{\widetilde{1}}{(\widetilde{1} + \widetilde{\mathcal{R}^f})}(\widetilde{\mathcal{P}}_{\mathcal{U}^{\uparrow}}\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}}) + \widetilde{\mathcal{P}}_{\mathcal{D}^{\downarrow}}\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{D}\mathcal{D}^{\downarrow}}))\right\}\\
&\approx (4.3476, 4.3989, 4.4504, 4.5339, 4.6390, 4.8454, 5.0150, 5.1521)\\
\widetilde{\mathcal{V}}_0^{\mathcal{A}}(\widetilde{\mathcal{F}}_0) &\approx \max\left\{\widetilde{K} - \widetilde{\mathcal{F}}_0, \frac{\widetilde{1}}{(\widetilde{1} + \widetilde{\mathcal{R}^f})}(\widetilde{\mathcal{P}}_{\mathcal{U}^{\uparrow}}\widetilde{\mathcal{V}}_1^{\mathcal{A}}(\widetilde{\mathcal{F}}_1^{\mathcal{U}^{\uparrow}}) + \widetilde{\mathcal{P}}_{\mathcal{D}^{\downarrow}}\widetilde{\mathcal{V}}_1^{\mathcal{A}}(\widetilde{\mathcal{F}}_1^{\mathcal{D}^{\downarrow}}))\right\}\\
&\approx (2.1801, 2.2540, 2.3394, 2.4630, 2.5807, 2.7756, 2.9339, 3.0674)
\end{aligned}$$

Defuzzification: Taking $k = 0.3$,, we obtain the defuzzified fuzzy future prices and American put fuzzy option prices as follows:

$$\begin{aligned}
M^{oct}(\widetilde{\mathcal{F}}_2^{\mathcal{U}\mathcal{U}^{\uparrow}}) &= \frac{1}{4}\{114.05751 + 266.21231\} = 95.0675\\
M^{oct}(\widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}}) &= \frac{1}{4}\{104.53281 + 243.90835\} = 87.1103\\
M^{oct}(\widetilde{\mathcal{F}}_2^{\mathcal{D}\mathcal{D}^{\downarrow}}) &= \frac{1}{4}\{95.80341 + 233.4729\} = 79.8191\\
M^{oct}(\widetilde{\mathcal{F}}_1^{\mathcal{U}^{\uparrow}}) &= \frac{1}{4}\{109.27839 + 255.0212\} = 91.0749\\
M^{oct}(\widetilde{\mathcal{F}}_1^{\mathcal{D}^{\downarrow}}) &= \frac{1}{4}\{100.15269 + 233.65475\} = 83.4519\\
M^{oct}(\widetilde{\mathcal{F}}_0) &= \frac{1}{4}\{104.69985 + 244.30063\} = 87.2501
\end{aligned}$$

$M^{oct}(\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}\mathcal{U}^{\uparrow}})) = 0; M^{oct}(\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}})) = 0.8897; M^{oct}(\widetilde{\mathcal{V}}_2^{\mathcal{A}}(\widetilde{\mathcal{F}}_2^{\mathcal{D}\mathcal{D}^{\downarrow}})) = 8.1809;$
$M^{oct}(\widetilde{_1^{\mathcal{A}}}(\widetilde{\mathcal{F}}_1^{\mathcal{U}^{\uparrow}})) = 0.4494; M^{oct}(\widetilde{\mathcal{V}}_1^{\mathcal{A}}(\widetilde{\mathcal{F}}_1^{\mathcal{D}^{\downarrow}})) = 4.6505; M^{oct}(\widetilde{\mathcal{V}}_0^{\mathcal{A}}(\widetilde{\mathcal{F}}_0)) = 2.5604.$

The fuzzy optimal expected prices at the nodes $\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}$, $\mathcal{D}\mathcal{D}^{\downarrow}$ and $\mathcal{D}^{\downarrow}$ are as follows:
$(85.452, 85.6278, 85.8364, 86.1152, 86.3256, 86.6038, 86.8148, 86.9908;)$
$(70.983, 71.1126, 71.3026, 71.4958, 71.6892, 72.0064, 72.2, 72.3938);$
$(78.1248, 78.2969, 78.5169, 78.7755, 78.9322, 79.0992, 79.2022, 79.3048).$
The corresponding crisp optimal expected prices and optimal exercise time of APFOM

involving non-overlapping octagonal and non-overlapping fuzzy trapezoidal numbers during the nodes $\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}$, $\mathcal{D}\mathcal{D}^{\downarrow}$ and $\mathcal{D}^{\downarrow}$ are represented in Tables 2 and 3 respectively.

Nodal points	Optimal - Exercise Time			- Expected reward
$\mathcal{U}\mathcal{U}^{\uparrow}$			Not exercise	
$\mathcal{U}^{\uparrow}\mathcal{D}^{\downarrow}$			exercise	86.2206
$\mathcal{D}\mathcal{D}^{\downarrow}$			exercise	71.6382
$\mathcal{D}^{\downarrow}$		exercise		78.8014
Time Period	Initial	Holding	Maturity	

TABLE 2. Optimal expected price using non-overlapping fuzzy octagonal numbers

Remark 6.1. When the American put fuzzy option problem is solved by using non-overlapping fuzzy trapezoidal numbers, we obtain the following fuzzy optimal expected prices
$(85.8364, 86.1152, 86.3256, 86.6038)$;
$(71.3026, 71.4958, 71.6892, 72.0064)$;
$(78.5169, 78.7755, 78.9322, 79.0992)$.
The corresponding crisp optimal expected prices at the nodes $\mathcal{U}\mathcal{D}$, $\mathcal{D}\mathcal{D}$ and $\mathcal{D}$ are

Nodes	Optimal - Exercise Time			- Expected price
$\mathcal{U}\mathcal{U}$			Not exercise	
$\mathcal{U}\mathcal{D}$			exercise	86.2203
$\mathcal{D}\mathcal{D}$			exercise	71.6235
$\mathcal{D}$		exercise		78.8310
time	Current	Retention	Expiry	

TABLE 3. Optimal expected price using non-overlapping fuzzy trapezoidal numbers

86.2203; 71.6235; 78.8310. As the future price of the stock is likely to go up during the node $\mathcal{U}\mathcal{D}$ at time $t = 2$, investor of the option can wait at the node $\mathcal{D}$ at time $t = 1$ and exercise the option during expiration. Thus, the option owner can get the optimal expected price involving non-overlapping fuzzy octagonal numbers than using non-overlapping fuzzy trapezoidal numbers.

7. Conclusion

We have modeled the jump factors, volatility parameter, and risk-free interest rate using non-overlapping fuzzy octagonal numbers to validate APFOM. From this, we have realized that the same yields optimal results.

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