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CONSTRUCTION OF DIFFERENTIAL IDENTITIES INVOLVING η-FUNCTIONS AND THE ASSESSMENT OF CONVOLUTION SUM

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Abstract. In his notebooks, Ramanujan recorded an extensive list of sophisticated correlations pertaining to the Eisenstein series. Shaun cooper mentioned various identities in his book, including Eisenstein series of level 10 and k-functions. In this article, we show a few application of these series in the invention of equations that involves class one infinite series and k-functions. Additionally, we generate numerous differential identities that involves k-functions. We supply an elementary method of determining a discrete convolution sum by utilizing relations including Eisenstein series and k-functions and assess a visually appealing representation for the discrete convolution sum.

1. Introduction

The combination of two signals to produce a new signal involves the utilization of a mathematical technique known as convolution. Input signal, output signal, and impulse response are three signals that are related through convolution. Numerous mathematical disciplines, including acoustics, spectroscopy, signal processing, image processing, geophysics, engineering, and physics, use convolution. As a practical example, we have computed the convolution sum of divisor functions, which exhibit a connection among Eisenstein series and k-functions.

Differential equations are versatile tools used across multiple disciplines to model and solve problems, and their application extends even to advanced mathematical areas like number theory and modular forms, where they can help uncover intricate relationships and properties among mathematical functions. Berndt et. al [3] stressed the significance of creating differential equations that include η-functions and Eisenstein series. This work inspired us to develop a few differential identities that link Eisenstein series with k-functions. A number of additional formulas involving k-functions that Ramanujan supplied, both on page 56 and in other random locations in his lost notebook [9]. He also represented the continued fractions of Roger's-Ramanujan in the context of k-functions. Ramanujan additionally generated formulas that link the Eisenstein series L, M, and N to the Class one infinite series $T_{2r}(q)$, $r = 1, 2, 6$ in his lost notebook [8]. In a recent development, Vidya H. C. et. al [10] formulated an expression for the Eisenstein

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series by utilizing the classical Class one infinite series. In this article, we utilize this formula to create fascinating equations connecting k-functions to the Class one infinite series. Ramanujan dedicated significant focus to both the development and utilization of Eisenstein series in his notebook [2] wherein he presented intriguing identities connecting these series with theta functions. Building upon this foundation, Vidya H. C. et. al [7] further expanded the study, formulating numerous differential equations and emphasizing their relevance in expressing k-functions interms of Class one infinite series. Drawing inspiration from their contributions, this article introduces several differential identities by applying Eisenstein relations at level 10.

Section 2 contains fundamental information and initial results that are necessary for understanding and accomplishing the main objectives of our research. In Section 3, we establish noteworthy first-order differential equations by leveraging Eisenstein series at level 10, incorporating Ramanujan's k-functions. Additionally, Section 4 explores the creation of novel expressions between Class one infinite series and Ramanujan's k-functions using identities that involve Eisenstein series and k-functions. Section 5 is devoted to the assessment of discrete convolution sums.

2. Preliminaries

2.1. Definition. Ramanujan provided the following definition of general theta function in his notebook [2]: Consider any complex a, b and q, with $|ab| < 1$,

$$\begin{aligned} f(a,b) &:= \sum_{i=-\infty}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \end{aligned}$$

where,

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad |q|<1.$$

The subsequent instances are specific cases of theta functions as defined by Ramanujan [2]:

$$\begin{aligned} \varphi(q) &:= f(q,q) = \sum_{i=-\infty}^{\infty} q^{i^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty, \\ \psi(q) &:= f(q,q^3) = \sum_{i=0}^{\infty} q^{i(i+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \\ f(-q) &:= f(-q,-q^2) = \sum_{i=-\infty}^{\infty}(-1)^i q^{i(3i-1)/2} \\ &= (q;q)_\infty = q^{-1/24}\eta(\tau), \end{aligned}$$

where $q = e^{2\pi i\tau}$.
We denote $f(-q^n) = f_n$.

2.2. Definition: [5] The Ramanujan function $k(q)$ in terms of the Roger's - Ramanujan continued fraction $r(q)$ is expressed as follows:

$$k(q) := r(q) \cdot r^2(q^2) = q \prod_{j=1}^{\infty} \frac{(1-q^{10j-9})(1-q^{10j-8})(1-q^{10j-2})(1-q^{10j-1})}{(1-q^{10j-7})(1-q^{10j-6})(1-q^{10j-4})(1-q^{10j-3})},$$

where

$$r(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \ldots$$

The Roger's-Ramanujan continued fraction is associated with the logarithmic derivatives of the Ramanujan function k, denoted as y_{10},

$$y_{10}(q) = q\frac{d}{dq}\log k.$$

2.3. Definition: [8] Ramanujan formulated an infinite series referred to as an Eisenstein series in the style of Ramanujan

$$P(q) := 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j} = 1 - 24\sum_{j=1}^{\infty} \sigma(j)q^j.$$

For simplicity, we denote $P(q^n) = P_n$.

2.4. Definition. [9] In his lost notebook, Ramanujan presented the Class one Infinite Series, which is formulated as follows:

$$T_{2l}(q) := 1 + \sum_{r=1}^{\infty}(-1)^r \left[(6r-1)^{2l}q^{r(3r-1)/2} + (6r+1)^{2l}q^{r(3r+1)/2}\right].$$

Ramanujan proceeded to represent the aforementioned infinite series using the Ramanujan-type Eisenstein series for values of l ranging from 1 to 6. Also, employing pentagonal number theorem [2], [3] established the correlation

$$\frac{T_2(q)}{(q;q)_\infty} = P(q).$$

Lemma 2.1 [5] There exists a connection among Eisenstein series and Ramanujan's k-function as expressed by the following:

$$\begin{bmatrix} P(q) \\ P(q^2) \\ P(q^5) \\ P(q^{10}) \end{bmatrix} = \begin{bmatrix} 4 & 1 & -4 & 6 \\ \frac{5}{2} & -2 & \frac{1}{2} & 3 \\ -\frac{4}{5} & 1 & \frac{4}{5} & \frac{6}{5} \\ \frac{1}{10} & \frac{2}{5} & \frac{1}{2} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1+k^2}{1-k^2}y_{10} \\ \frac{1+k^2}{1+k-k^2}y_{10} \\ \frac{1+k^2}{1-4k-k^2}y_{10} \\ k\frac{dy_{10}}{dk} \end{bmatrix}.$$

Lemma 2.2 [10] For every natural number n such that $n \geq 2$, the following relationship exists between the Eisenstein series and the Class one infinite series:

$$P(q^n) = 1 + nq^{(n-1)} \left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty} - 1 \right].$$

Lemma 2.3 [5] The relationship between the infinite series and the k-function is expressed as follows:

$$\text{(i)} \quad P_1 - 2P_2 - 5P_5 + 30P_{10} = 12\left(\frac{u(2u^2-5v-10)}{v(v+1)(v-4)}y_{10} + w\right),$$

$$\text{(ii)} \quad -P_1 + 6P_2 + 5P_5 - 10P_{10} = 12\left(\frac{3v-2}{v(v+1)(v-4)}y_{10} + w\right),$$

$$\text{(iii)} \quad -P_1 + 2P_2 + 15P_5 - 10P_{10} = 6\left(\frac{u(u^2-4v-8)}{v(v+1)(v-4)} - \frac{w}{2}\right),$$

$$\text{(iv)} \quad 3P_1 - 2P_2 - 5P_5 + 10P_{10} = 6\left(\frac{u(u^2-12v-12)}{v(v+1)(v-4)}y_{10} + w\right),$$

where $k + 1/k = u$, $-k + 1/k = v$, and $k\frac{dy_{10}}{dk} = w$.

Proof. The result holds true using Lemma 2.1 and further simplification. □

3. CONSTRUCTION OF DIFFERENTIAL EQUATIONS

Theorem 3.1. *If*

$$H(q) = \frac{\eta_1\eta_{10}^2}{\eta_2^2\eta_5},$$

then the subsequent differential relation holds:

$$q\frac{dH}{dq} - \left[\frac{u}{(1-v)}y_{10}\right]H = 0,$$

where $k + \frac{1}{k} = u$, $-k + \frac{1}{k} = v$.

Proof. Applying definition of theta function on $H(q)$, we get

$$H(q) = \frac{\eta_1\eta_{10}^2}{\eta_2^2\eta_5} = q^{1/2}\frac{f_1f_{10}^2}{f_2^2f_5}.$$

Now expressing in terms of q-series and taking logarithm on both sides and differentiating the resulting relation with respect to q , we deduce the subsequent relation

$$\frac{q}{H}\frac{dH}{dq} = \frac{1}{2} - \sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} + 8\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}} + 5\sum_{r=1}^{\infty}\frac{rq^{5r}}{1-q^{5r}} - 20\sum_{r=1}^{\infty}\frac{rq^{10r}}{1-q^{10r}}.$$

By applying the Eisenstein series definition and simplifying, we arrive at the subsequent relationship:

$$\frac{q}{H}\frac{dH}{dq} = \frac{1}{24}\left(P_1 - 4P_2 - 5P_5 + 20P_{10}\right).$$

Eventually, we figure out the necessary identity by using Lemma 2.1. □

Theorem 3.2. *If*

$$H(q) = \frac{f_1 f_{10}^5}{f_2 f_5^5},$$

then the subsequent differential relation holds:

$$q\frac{dH}{dq} - \left[\frac{u}{v} y_{10}\right] H = 0,$$

where $k + \frac{1}{k} = u$, $-k + \frac{1}{k} = v$.

Proof. Using definition theta function, q-series in $H(q)$ and taking logarithm on both sides and differentiating the resulting relationship with respect to q, we deduce

$$\frac{q}{H}\frac{dH}{dq} = 1 - \sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} + 2\sum_{r=1}^{\infty} \frac{rq^{2r}}{1-q^{2r}} + 25\sum_{r=1}^{\infty} \frac{rq^{5r}}{1-q^{5r}} - 50\sum_{r=1}^{\infty} \frac{rq^{10r}}{1-q^{10r}}.$$

By streamlining and expanding using the Eisenstein series , we achieve at

$$\frac{q}{H}\frac{dH}{dq} = \frac{1}{24}\left(P_1 - 2P_2 - 25P_5 + 50P_{10}\right).$$

Finally, Lemma 2.1 is used to determine the necessary differential equation. □

Theorem 3.3. *If*

$$H(q) = \frac{f_2 f_{10}^3}{f_1^3 f_5},$$

then the resulting differential equation holds:

$$q\frac{dH}{dq} - \left[\frac{u}{(v-4)} y_{10}\right] H = 0,$$

where $k + \frac{1}{k} = u$, $-k + \frac{1}{k} = v$.

Proof. By representing the theta function $H(q)$ in the form of q-series, subsequently applying logarithmic transformations to both sides, differentiating the resulting relation with respect to q, and further incorporating the definition of Eisenstein series, we derive

$$\frac{q}{H}\frac{dH}{dq} = 1 + 3\sum_{r=1}^{\infty} \frac{rq^r}{1-q^r} - 2\sum_{r=1}^{\infty} \frac{rq^{2r}}{1-q^{2r}} + 5\sum_{r=1}^{\infty} \frac{rq^{5r}}{1-q^{5r}} - 30\sum_{r=1}^{\infty} \frac{rq^{10r}}{1-q^{10r}}.$$

We calculate the conclusion using lemma 2.1. □

Theorem 3.4. *If*

$$H(q) = \frac{f_1 f_2^2 f_5^3}{f_{10}^2},$$

consequently, the differential identity is valid:

$$q\frac{dH}{dq} - wH = 0,$$

where $k + \frac{1}{k} = u$, $-k + \frac{1}{k} = v$, *and* $k\frac{dy_{10}}{dk} = w$.

Proof. Employing the definition theta function, q-series and taking logarithm on either sides and further differentiating the resulting relation with respect to q, we deduce

$$q\frac{dH}{dq} = 3\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - 2\sum_{r=1}^{\infty}\frac{rq^{2r}}{1-q^{2r}} + 5\sum_{r=1}^{\infty}\frac{rq^{5r}}{1-q^{5r}} - 30\sum_{r=1}^{\infty}\frac{rq^{10r}}{1-q^{10r}}.$$

We determine the necessary differential equation by using Lemma 2.1 in conjunction with the definition of the Eisenstein series. □

4. CONNECTION AMONG CLASS ONE INFINITE SERIES AND K-FUNCTIONS

Theorem 4.1. *The following series expression among Class one infinite series and k – function holds:*

(1) $$\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} - 25q^4\frac{T_2(q^5)}{f_5} + 300q^9\frac{T_2(q^{10})}{f_{10}} + 4q\left(1-\frac{1}{f_2}\right) + 25q^4\left(1-\frac{1}{f_5}\right)$$
$$-300q^9\left(1-\frac{1}{f_{10}}\right) - 12\left(\frac{u(2u^2-5v-10)}{v(v+1)(v-4)}y_{10} + w\right) + 23 = 0$$

(2) $$-\frac{T_2(q)}{f_1} + 12q\frac{T_2(q^2)}{f_2} + 25q^4\frac{T_2(q^5)}{f_5} - 100q^9\frac{T_2(q^{10})}{f_{10}} - 12q\left(1-\frac{1}{f_2}\right) - 25q^4\left(1-\frac{1}{f_5}\right)$$
$$+100q^9\left(1-\frac{1}{f_{10}}\right) - 12\left(\frac{3v-2}{v(v+1)(v-4)}y_{10} + w\right) + 1 = 0$$

(3) $$-\frac{T_2(q)}{f_1} + 4q\frac{T_2(q^2)}{f_2} + 75q^4\frac{T_2(q^5)}{f_5} - 100q^9\frac{T_2(q^{10})}{f_{10}} - 4q\left(1-\frac{1}{f_2}\right) - 75q^4\left(1-\frac{1}{f_5}\right)$$
$$+100q^9\left(1-\frac{1}{f_{10}}\right) - 6\left(\frac{u(u^2-4v-8)}{v(v+1)(v-4)}y_{10} - \frac{w}{2}\right) + 7 = 0$$

(4) $$3\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} - 25q^4\frac{T_2(q^5)}{f_5} + 100q^9\frac{T_2(q^{10})}{f_{10}} + 4q\left(1-\frac{1}{f_2}\right) + 25q^4\left(1-\frac{1}{f_5}\right)$$
$$-100q^9\left(1-\frac{1}{f_{10}}\right) - 6\left(\frac{u(u^2-12v-12)}{v(v+1)(v-4)}y_{10} + w\right) + 3 = 0$$

where $k+\frac{1}{k}=u$, $-k+\frac{1}{k}=v$, *and* $k\frac{dy_{10}}{dk}=w$.

Proof. We express Lemma 2.3 (i) to (iv) in terms of $T_2(q^n)$ for $n = 1, 2, 5, 10$ by employing Lemma 2.2. Hence the proof. □

5. EVALUALTION OF CONVOLUTION SUM

In this section, we provide a simple method of evaluating a discrete convolution sum. We begin with the definition of Convolution Sum. For $k, n \in \mathbb{N}$, we set $\delta_k(n) = \sum_{d|n} d^k$, where d runs through the positive integer's divisors of n. If $n \notin \mathbb{N}$, we set $\delta_k(n) = 0$. For $i, j \in \mathbb{N}$ with $i \leq j$, the convolution sum

$$W_{i,j}(n) = \sum_{ni+mj=n} \sigma(i)\sigma(j).$$

Alaca, et al. [1] have explicitly computed the convolution $W_{i,j}(n) = \sum_{ni+mj=n} \sigma(i)\sigma(j)$ for various values of i and j across all n. Wonderful introduction can be found in Alaca et. al [1]. The article by Xia et. al [11] provides representations for the convolution $\sum_{i+6j=n} \sigma(i)\sigma(j)$ and $\sum_{i+12j=n} \sigma(i)\sigma(j)$. The proofs for our results align with the assertions made by Glaisher [6],

$$P^2(q) = 1 + \sum_{n=1}^{\infty}(240\sigma_3(n) - 288\sigma(n))q^n. - [1]$$

Theorem 5.1. *For any $n \in \mathbb{N}$, we have the convolution sum,*

$$(i) \sum_{i+2j=n} \sigma(i)\sigma(j) = \frac{1}{24}\Bigg[(1-6n)\sigma(n) + (1-3n)\sigma\left(\frac{n}{2}\right) + 5(\sigma_3(n) + \sigma_3\left(\frac{n}{2}\right)) - \frac{1}{48}A(k)\Bigg],$$

$$(ii) \sum_{i+5j=n} \sigma(i)\sigma(j) = \frac{1}{24}\Bigg[(1-6n)\sigma(n) + \left(1-\frac{6}{5}n\right)\sigma\left(\frac{n}{5}\right) + 5(\sigma_3(n) + \sigma_3\left(\frac{n}{2}\right)) - \frac{1}{48}B(k)\Bigg],$$

$$(iii) \sum_{i+10j=n} \sigma(i)\sigma(j) = \frac{1}{24}\Bigg[(1-8n)\sigma(n) + \left(1-\frac{3}{5}n\right)\sigma\left(\frac{n}{10}\right) + 5(\sigma_3(n) + \sigma_3\left(\frac{n}{10}\right)) - \frac{1}{48}C(k)\Bigg],$$

$$(iv) \sum_{2i+5j=n} \sigma(i)\sigma(j) = \frac{1}{24}\Bigg[(1-3n)\sigma\left(\frac{n}{2}\right) + \left(1-\frac{6}{5}n\right)\sigma\left(\frac{n}{5}\right) + 5(\sigma_3\left(\frac{n}{2}\right) + \sigma_3\left(\frac{n}{5}\right)) - \frac{1}{48}D(k)\Bigg],$$

where

$$\sum_{n=1}^{\infty} A(k)q^n = \left(\frac{1}{2}\left[-\frac{u(10u+4)}{v(v+1)(v-4)}y_{10} + 9w\right]\right)^2,$$

$$\sum_{n=1}^{\infty} B(k)q^n = \left(\frac{24}{5}\left[-\frac{4u}{v(v-4)}y_{10} + w\right]\right)^2,$$

$$\sum_{n=1}^{\infty} C(k)q^n = \left(\frac{3}{10}\left[\frac{u(62v-52)}{v(v+1)(v-4)}y_{10} + \frac{33}{5}w\right]\right)^2,$$

$$\sum_{n=1}^{\infty} D(k)q^n = \left(\frac{3}{10}\left[\frac{u(6v-44)}{v(v+1)(v-4)}y_{10} + \frac{9}{5}w\right]\right)^2,$$

$k+\frac{1}{k}=u,\ -k+\frac{1}{k}=v,\ and\ k\frac{dy_{10}}{dk}=w.$

Proof. Using Lemma 2.1, we derive

$$P(q)-P(q^2)=\frac{3}{2}\left[\frac{1+k^2}{1-k^2}\right]-\left[\frac{1+k^2}{1+k-k^2}\right]-\frac{7}{2}\left[\frac{1+k^2}{1-4k-k^2}\right]y_{10}+9w,$$

$$P(q)-P(q^5)=\frac{24}{5}\left[\frac{1+k^2}{1-k^2}\right]-\frac{24}{5}\left[\frac{1+k^2}{1-4k-k^2}\right]y_{10}+\frac{24}{5}w,$$

$$P(q)-P(q^{10})=\frac{39}{10}\left[\frac{1+k^2}{1-k^2}\right]+\frac{3}{5}\left[\frac{1+k^2}{1+k-k^2}\right]-\frac{9}{2}\left[\frac{1+k^2}{1-4k-k^2}\right]y_{10}+\frac{33}{5}w,$$

$$P(q^2)-P(q^5)=\frac{33}{10}\left[\frac{1+k^2}{1-k^2}\right]-2\left[\frac{1+k^2}{1+k-k^2}\right]-\frac{3}{10}\left[\frac{1+k^2}{1-4k-k^2}\right]y_{10}+9w,$$

where $k\frac{dy_{10}}{dk}=w$.
We acquire the compound convolution sum by squaring these results on either side, using the Eisenstein series formulation, and further adopting relation [1] . □

6. CONCLUSION

In advanced mathematical areas like number theory and modular forms, differential equations can uncover deep properties and relationships among mathematical functions. The versatility and wide applicability of differential equations make them indispensable tools in both theoretical and applied mathematics. In this article, we have constructed beautiful differential relations that include k-functions. Convolution is a technique that can be used in any scientific discipline that requires the computation of mathematical data in the form of multiplying and then accumulating. The most popular methods used by mathematicians to determine convolution sums include Ramanujan's discriminant function, Gaussian hyper-geometric series, quasi-modular forms, Ramanujan-type Eisenstein series, and many others. Convolution finds extensive applications in various domains, including numerical linear algebra, probability theory, numerical analysis, deep learning, and the design and implementation of finite impulse response filters in signal processing. Inspired by this, in this article, the convolution sum was carried out, which helps to compute the discrete convolution of two infinite products.

References

1. Alaca A., Alaca S. and Williams K. S.: Evaluation of the convolution sums $\sum_{l+12m=n}\sigma(l)\sigma(m)$ and $\sum_{3l+4m=n}\sigma(l)\sigma(m)$, *Journal of Number Theory,* **124**(2)(2007) 491-510.
2. Berndt B. C.: *Ramanujan's Notebooks*, Part III, Springer, New York, 1991.
3. Berndt B. C., and Andrews G. E.: Series Representable in terms of Eisenstein series, *Ramanujan's Lost Notebook,* Springer, New York, 2009.
4. Borwein J. M. and Borwein, P. B.: A cubic counterpart of Jacobi's identity and the AGM, *Trans. Am. Math. Soc*, **323** (1991) 691-701.
5. Cooper S.: *Ramanujan's Theta Functions*, Springer, 2017.
6. Glaisher J. W. L.: On the square of the series in which the coeff's are the sum of the divisors of the exponent, *Messenger of Mathematics*, **14**(1885) 156-163.
7. Harekala Chandrashekara Vidya and Badanidiyoor Ashwath Rao.: Intriguing Relationships among Eisenstein Series, Borewein's Cubic Theta Functions, and the Class One Infinite Series, *IAENG International Journal of Computer Science,* **50**(4) (2023) 1166-1173.

8. Ramanujan, S.: *Notebooks, Vol. 1,2*, Tata Institute of Fundamental Research, Mumbai, India, 1957.
9. Ramanujan S.: *The Lost Notebook and Other Unpublished Papers,* Narosa, New Delhi, 1988.
10. Vidya H. C , Pragathi B Shetty and Ashwath Rao B.: Formulation of differential equations utilizing the relationship among Ramanujan-type Eisenstein series and h-functions, *Global and Stochastic Analysis,* **11**(3) (2024) 21-31.
11. Xia E. X. W. and Yao O. X. M.: Eisenstein series identities involving the Borwein's cubic theta functions, *Journal of Applied Mathematics*, Article 181264 , 2012.

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